Falling Water Tables in Horizontal/Sloping Aquifer

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Abstract: To describe falling water tables between two drains lying on a horizontal/sloping impermeable barrier, analytical solutions of the Boussinesq equation linearized by Baumann's and Werner's methods and numerical solutions of the nonlinear form of the Boussinesq equation using finite-difference and finite-element methods were obtained. A hybrid finite analytic method, in which the nonlinear Boussinesq equation was locally linearized and solved analytically after approximating the unsteady term by a simple finite-difference formula to approximately preserve the overall nonlinear effect by the assembly of locally analytic solutions, was also used to obtain a solution of the Boussinesq equation. Midpoints of falling water tables between two drains in a horizontal/sloping aquifer as obtained from various solutions were compared with already existing experimental values. Euclidean $L2$ and Tchebycheff L_∞ norms were used to rank the performance of various solutions with respect to experimental data. It was observed that the performance of the hybrid finite analytic solution is the best, followed by finite element, finite difference, analytical with Werner's linearization method, and analytical with Baumann's linearization method, respectively.

INTRODUCTION

Most of the subsurface drainage theories related to flatlands or moderately sloping lands have been developed by obtaining the solution of partial differential equations derived by Boussinesq (1877, 1904), based on the principle of continuity and Dupuit-Forchheimer assumptions. Many investigators, such as Schmid and Luthin (1964), Guitjens and Luthin (1965), Chauhan et al. (1968), Childs (1971), Towner (1975), Jaiswal and Chauhan (1975), Singh and Jacob (1977), Chapman (1980), Sloan and Moore (1984), Yates et al. (1985), Sewa Ram and Chauhan (1987a,b), Fipps and Skaggs (1989), Shukla et al. (1990, 1999), Sanford et al. (1993), Brutsaert (1994), Pi and Hjelmfelt (1994), Kalaidzidou-Paikou et al. (1997), Koussis et al. (1998), Connell et al. (1998), and Steenhuis et al. (1999), have obtained analytical, numerical, or experimental solutions of linearized or nonlinear forms of the Boussinesq continuity equation to describe spatial and temporal variation of water tables in an aquifer resting on a sloping impermeable barrier. A brief review of these studies has been presented by Upadhyaya (1999) and Upadhyaya and Chauhan (2000). Most of the analytical solutions obtained are approximate because of the linearization of the governing equation. Numerical solutions of nonlinear equations are better than the analytical solutions of linearized equations. However, the accuracy of such solutions can be tested by comparing the results with experimental values considering the latter ones as the benchmark solution. The objective of this paper is to obtain analytical solutions of the linearized Boussinesq equation and numerical solutions (based on finite-difference, finite-element, and hybrid finite analytic techniques) of the nonlinear Boussinesq equation to describe falling water tables between two drains in a horizontal or moderately sloping aquifer and comparison of predicted water table elevations with the experimental results obtained through studies of Chauhan (1967) and Chauhan et al. (1968) simulating falling water tables for the horizontal and sloping cases on a vertical Hele-Shaw model.

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BOUNDARY-VALUE PROBLEM AND SOLUTIONS

While formulating the boundary-value problem for falling water tables between two conventional parallel drains, it is assumed that due to instantaneous recharge the water table has reached the land surface and it falls because of drainage of the aquifer. The nonlinear second-order partial differential equation derived by Boussinesq (1904) to describe falling water tables between two drains located in a sloping aquifer may be written

$$h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 - \alpha\left(\frac{\partial h}{\partial x}\right) = \frac{f}{K}\frac{\partial h}{\partial t} \quad (1a)$$

and for a horizontal aquifer (considering $\alpha = 0$)

$$h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 = \frac{f}{K}\frac{\partial h}{\partial t} \quad (1b)$$

Here h = height of water table above the impermeable barrier (L) at a distance x and time t; α = slope of the impermeable barrier having a small value such that $\alpha = \sin\alpha = \tan\alpha$; K = hydraulic conductivity (L T^{-1}); and f = drainable porosity of the aquifer.

Two techniques may be used to linearize the nonlinear equations [(1a) and (1b)]. The first technique, known as Baumann's technique takes $h(x, t) = D + \varepsilon(x, t)$, where D is a characteristic depth and $\varepsilon << D$, and by neglecting higher order in ε, eliminates $(\partial h/\partial x)^2$. The second technique is based on Werner's transformation, $z = h^2$, and linearizes equations by setting $(1/\sqrt{z})\cdot(\partial z/\partial t) = (1/D)\cdot(\partial z/\partial t)$. Here the characteristic depth $D = h_0/2$ or average depth of flow and h_0 is initial water table height above the impermeable barrier. After applying the above linearization techniques, (1a) corresponding to a sloping aquifer can be expressed as (2a) and (3a) whereas (1b) corresponding to a horizontal aquifer can be expressed as (2b) and (3b) as below

$$\frac{\partial^2 h}{\partial x^2} - 2s\left(\frac{\partial h}{\partial x}\right) = \frac{1}{a}\frac{\partial h}{\partial t};\quad \frac{\partial^2 h}{\partial x^2} = \frac{1}{a}\frac{\partial h}{\partial t} \quad (2a,b)$$

$$\frac{\partial^2 z}{\partial x^2} - 2s\left(\frac{\partial z}{\partial x}\right) = \frac{1}{a}\frac{\partial z}{\partial t};\quad \frac{\partial^2 z}{\partial x^2} = \frac{1}{a}\frac{\partial z}{\partial t} \quad (3a,b)$$

Here $a = KD/f$; and $s = \alpha/2D$. Initial shape of the water table may be assumed flat, parabolic, or elliptical, depending on soil characteristics. In this study, a flat initial water table near the land surface and zero water table at the drains (neglecting the effect of a seepage surface) have been considered. The definition sketch of the flow problem is given in Fig. 1. The initial and boundary conditions in mathematical terms corresponding to (1a), (1b), (2a), and (2b) may be written

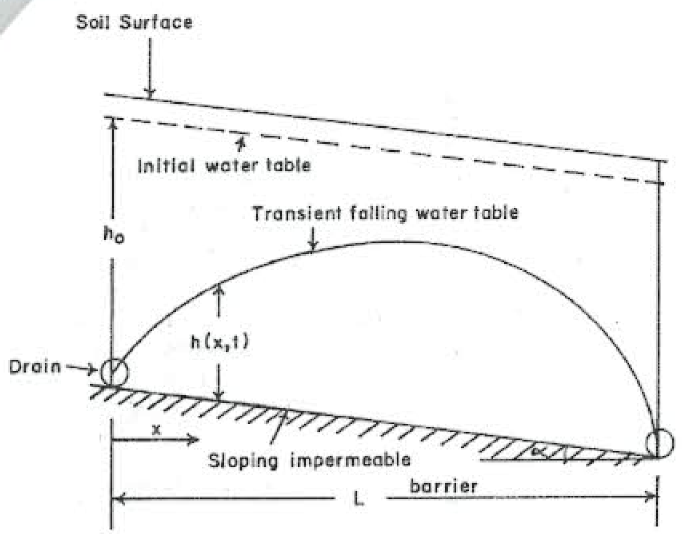

FIG. 1. Definition Sketch of Falling Water Table in Sloping Aquifer

$$h(x, 0) = h_0 \quad \text{at } t = 0 \quad \text{for } 0 < x < L \tag{4a}$$

$$h(0, t) = h(L, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = 0 \text{ and } x = L \tag{4b}$$

whereas, corresponding to (3*a*) and (3*b*), these conditions are

$$z(x, 0) = h_0^2 = z_0 \quad \text{at } t = 0 \quad \text{for } 0 < x < L \tag{5a}$$

$$z(0, t) = z(L, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = 0 \text{ and } x = L \tag{5b}$$

Here h_0 represents initial water table height above the impermeable barrier.

Analytical Solutions

An analytical solution of linearized Boussinesq equation (2*a*) with initial and boundary conditions (4*a*) and (4*b*) was obtained by devising a transformation that absorbs the term associated with slope and converts (2*a*) into a simple heat flow equation. The transformation may be given

$$h = v e^{sx - s^2 at} \tag{6}$$

With this transformation the governing equation (2*a*) becomes

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{a}\frac{\partial v}{\partial t} \tag{7a}$$

and the initial and boundary conditions

$$v(x, 0) = h_0 e^{-sx} = f(x) \quad \text{at } t = 0 \quad \text{for } 0 < x < L \tag{7b}$$

$$v(0, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = 0 \tag{7c}$$

$$v(L, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = L \tag{7d}$$

The solution of (7*a*) with initial and boundary conditions (7*b*)–(7*d*) is given by Ozisik (1980)

$$v(x, t) = \frac{2}{L}\sum_{m=1}^{\infty} e^{-a\beta_m^2 t} \sin \beta_m x \int_0^L f(x) \sin \beta_m x' \, dx' \tag{8}$$

with $\beta_m = m\pi/L$. Substitution of the value of $f(x)$ from (7*b*) in (8) and applying the inverse transformation gives the final solution

$$h(x, t) = \frac{2h_0}{L} e^{sx - s^2 at} \sum_{m=1}^{\infty} e^{-a\beta_m^2 t} \sin \beta_m x \left\{ \frac{(1 - (-1)^m e^{-sL})\beta_m}{(s^2 + \beta_m^2)} \right\} \tag{9}$$

If the aquifer is considered horizontal, the boundary-value problem is defined by (2*b*), (4*a*), and (4*b*) and the analytical solution may be obtained by putting $s = 0$ in (9)

$$h(x, t) = \frac{4h_0}{L} \sum_{m=1,3,5,\ldots}^{\infty} e^{-a\beta_m^2 t} \frac{\sin \beta_m x}{\beta_m} \tag{10}$$

This solution is similar to the solution proposed by Dumm (1954).

Analytical solutions of Boussinesq's equation linearized by Werner's transformation as given by (3*a*) and (3*b*) with initial and boundary conditions (5*a*) and (5*b*) were obtained using the same technique mentioned above. The transformation employed to (3*a*) is similar to (6) except that, in place of h, z is used. The transformed initial and boundary conditions are also the same as (7*b*)–(7*d*) except that in (7*b*), in place of h_0, z_0 is used. The analytical solution for falling water tables in a sloping aquifer in the form of $z(x, t)$ may be expressed

$$z(x, t) = \frac{2z_0}{L} e^{sx - s^2 at} \sum_{m=1}^{\infty} e^{-a\beta_m^2 t} \sin \beta_m x \left\{ \frac{(1 - (-1)^m e^{-sL})\beta_m}{(s^2 + \beta_m^2)} \right\} \tag{11}$$

and in case of a horizontal aquifer, corresponding to (3*b*), the analytical solution may be written

$$z(x, t) = \frac{4z_0}{L} \sum_{m=1,3,5,\ldots}^{\infty} e^{-a\beta_m^2 t} \frac{\sin \beta_m x}{\beta_m} \tag{12}$$

The solutions expressed by (11) and (12) are in the form of $z(x, t)$. To obtain the expression for $h(x, t)$, the square root of the expression for $z(x, t)$ is used.

Finite-Difference Solution

To obtain a finite-difference solution of nonlinear Boussinesq equation (1*a*) along with initial and boundary conditions given by (4*a*) and (4*b*), (1*a*) is made dimensionless with the help of a set of variables, $H = h/h_0$, $X = x/L$, and $T = Kh_0 t/fL^2$. After transforming (1*a*) with these dimensionless variables, the governing equation and initial and boundary conditions may be written

$$H\frac{\partial^2 H}{\partial X^2} + \left(\frac{\partial H}{\partial X}\right)^2 - A\left(\frac{\partial H}{\partial X}\right) = \frac{\partial H}{\partial T} \tag{13}$$

or

$$\frac{1}{2}\frac{\partial^2 H^2}{\partial X^2} - A\left(\frac{\partial H}{\partial X}\right) = \frac{\partial H}{\partial T} \tag{14}$$

$$H(X, 0) = 1 \quad \text{at } T = 0 \quad \text{for } 0 < X < 1 \tag{15a}$$

$$H(0, T) = H(1, T) = 0 \quad \text{at } T > 0 \quad \text{for } X = 0 \text{ and } X = 1 \tag{15b}$$

where $A = \alpha L/h_0$.

Eq. (14) can be discretized in finite-difference form as below

$$\begin{aligned} \frac{H_m^{n+1} - H_m^n}{\Delta T} &= \frac{1}{2(\Delta X)^2}[\theta(H_{m-1}^{n+1})^2 + (1-\theta)(H_{m-1}^n)^2 - 2\theta(H_m^{n+1})^2 \\ &\quad - 2(1-\theta)(H_m^n)^2 + \theta(H_{m+1}^{n+1})^2 + (1-\theta)(H_{m+1}^n)^2] \\ &\quad - \frac{A}{2\Delta X}[\theta(H_{m+1}^{n+1}) + (1-\theta)(H_{m+1}^n) - \theta(H_{m-1}^{n+1}) - (1-\theta)(H_{m-1}^n)] \end{aligned} \tag{16}$$

Here θ may be assigned a value from 0 to 1, resulting in explicit or implicit finite-difference schemes. Subscript m denotes a variable in the space grid and subscript n denotes a variable in time.

Jain et al. (1994) proposed the following procedure to solve the system of nonlinear equations. Let $H_m^{n+1} = H_m^n + V_m^n$. Substituting this in the above equation one gets

$$V_m^n = \frac{\Delta T}{2(\Delta X)^2}[\theta(H_{m-1}^n + V_{m-1}^n)^2 + (1-\theta)(H_{m-1}^n)^2 - 2\theta(H_m^n + V_m^n)^2$$

$$- 2(1-\theta)(H_m^n)^2 + \theta(H_{m+1}^n + V_{m+1}^n)^2 + (1-\theta)(H_{m+1}^n)^2]$$

$$- \frac{A\Delta T}{2\Delta X}[\theta(H_{m+1}^n + V_{m+1}^n) + (1-\theta)(H_{m+1}^n) - \theta(H_{m-1}^n + V_{m-1}^n)$$

$$- (1-\theta)(H_{m-1}^n)] \quad (17)$$

Keeping $(\Delta T/\Delta X) = C$ and $(\Delta T/(\Delta X)^2) = \lambda$ and neglecting the terms of the order of $O(V^2)$, one gets the following equation:

$$V_{m-1}^n\left[\lambda\theta H_{m-1}^n + \frac{AC\theta}{2}\right] + V_m^n[-2\lambda\theta H_m^n - 1]$$

$$+ V_{m+1}^n\left[\lambda\theta H_{m+1}^n - \frac{AC\theta}{2}\right] = -\frac{\lambda}{2}[(H_{m-1}^n)^2 - 2(H_m^n)^2$$

$$+ (H_{m+1}^n)^2] + \frac{AC}{2}[H_{m+1}^n - H_{m-1}^n] \quad (18)$$

By inputting different values of θ, one can obtain various finite-difference schemes.

If $\theta = 0$, one can get an explicit finite-difference scheme

$$V_m^n = \frac{\lambda}{2}[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] - \frac{AC}{2}[H_{m+1}^n - H_{m-1}^n] \quad (19a)$$

If $\theta = 0.5$, the Crank-Nicolson finite-difference scheme may be written

$$V_{m-1}^n\left[\lambda H_{m-1}^n + \frac{AC}{2}\right] - 2V_m^n[\lambda H_m^n + 1] + V_{m+1}^n\left[\lambda H_{m+1}^n - \frac{AC}{2}\right]$$

$$= -\lambda[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] + AC[H_{m+1}^n - H_{m-1}^n] \quad (19b)$$

and if $\theta = 1.0$, one can obtain a fully implicit finite-difference scheme

$$V_{m-1}^n\left[\lambda H_{m-1}^n + \frac{AC}{2}\right] - V_m^n[2\lambda H_m^n + 1] + V_{m+1}^n\left[\lambda H_{m+1}^n - \frac{AC}{2}\right]$$

$$= -\frac{\lambda}{2}[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] + \frac{AC}{2}[H_{m+1}^n - H_{m-1}^n] \quad (19c)$$

By putting $A = 0$ in (19a)–(19c), the expression of the finite-difference solution for falling water tables between two conventional level drains in a horizontal aquifer can be obtained.

This system of algebraic equations formed at a given time step is a tridiagonal matrix for which a solution can be obtained by any of the standard algorithms available in various texts of numerical analysis and V_{m-1}^n, V_m^n, and V_{m+1}^n can be computed. To get the values at $n + 1$ time step (i.e., H_{m-1}^{n+1}, H_m^{n+1}, and H_{m+1}^{n+1}), the values of H_{m-1}^n, H_m^n, and H_{m+1}^n are added into V_{m-1}^n, V_m^n, and V_{m+1}^n, respectively.

Finite-Element Solution

A finite-element solution of the dimensionless nonlinear Boussinesq equation as shown by (13) or (14) along with initial and boundary conditions (15a) and (15b) was obtained using Galerkin's method, the details of which are given in Pinder and Gray (1977). The flow domain is discretized as $0 = X_1 < X_2 < X_3 < X_4 < \cdots < X_{N-1} < X_N = 1$. Here N represents the number of nodes and $\Delta X = X_{i+1} - X_i$, where $i = 1, 2, 3, \ldots, N - 1 = M$, the number of elements.

To obtain the solution of Boussinesq's equation by Galerkin's method, a linear Lagrange polynomial is associated with each node. A typical basis function associated with each node X_i, as defined by Prenter (1975) is given below

$$N_i(X) = \frac{(X - X_{i-1})}{(X_i - X_{i-1})} \quad \text{for } X_{i-1} \le X \le X_i \quad (20a)$$

$$N_i(X) = \frac{(X_{i+1} - X)}{(X_{i+1} - X_i)} \quad \text{for } X_i \le X \le X_{i+1} \quad (20b)$$

The basis function $N_i(X)$ has the value unity at the node with which it is associated and is zero at all the other nodes. The basis function $N_i(X)$ has a component in each of the two elements joining at node X_i. Hence, over the elements (X_{i-1}, X_i) and (X_i, X_{i+1}) there are two nonzero basis functions $N_{i-1}(X)$, $N_i(X)$ and $N_i(X)$, $N_{i+1}(X)$. The basis function $N_i(X)$ over the elements (X_{i-1}, X_i) and (X_i, X_{i+1}) has already been defined by (20a) and (20b). The basis functions $N_{i-1}(X)$ over the element (X_{i-1}, X_i) and $N_{i+1}(X)$ over the element (X_i, X_{i+1}) are given below

$$N_{i-1}(X) = \frac{(X_i - X)}{(X_i - X_{i-1})} \quad \text{for } X_{i-1} \le X \le X_i \quad (20c)$$

$$N_{i+1}(X) = \frac{(X - X_i)}{(X_{i+1} - X_i)} \quad \text{for } X_i \le X \le X_{i+1} \quad (20d)$$

The value of all other basis functions are zero over the elements (X_{i-1}, X_i) and (X_i, X_{i+1}). The solution is approximated by $H^A(X, T)$ with the help of the basis functions as follows:

$$H^A(X, T) = \sum_{i=1}^{N} Z_i(T)N_i(X) \quad (21)$$

in which $Z_i(T)$ = unknown coefficients to be determined as a part of the solution. The multiplier $Z_i(T)$ associated with $N_i(X)$ at node i is the value of H at i. Because there are only two nonzero basis functions over an element (X_i, X_{i+1}), the summation is performed only over two consecutive indices, i and $i + 1$ to approximate the solution $H^A(X, T)$ over the element.

To carry out finite-element analysis, (13) or (14) may be written

$$L(H) = \frac{\partial}{\partial X}\left(H\frac{\partial H}{\partial X}\right) - A\left(\frac{\partial H}{\partial X}\right) - \frac{\partial H}{\partial T} = 0 \quad (22)$$

The expression $H^A(X, T)$ is an approximation for $H(X, T)$. Hence its substitution in (22) leaves a residual $L(H^A)$, which is used to determine the coefficients $Z_i(T)$. As there are N unknown coefficients to be determined, N constraints have to be imposed on the residual $L(H^A)$ to evaluate these coefficients. In Galerkin's finite-element method, the coefficients $Z_i(T)$ are determined by forcing the residual $L(H^A)$ to be orthogonal to the basis functions $N_i(X)$, where $i = 1, 2, 3, \ldots, N$. For this, the inner product of $L(H^A)$ with $N_i(X)$ has to be zero; i.e.

$$\langle L(H^A)\cdot N_i(X)\rangle = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \quad (23)$$

Substitution of (22) in (23) yields

$$\left\langle\frac{\partial}{\partial X}\left(H\frac{\partial H}{\partial X}\right), N_i(X)\right\rangle - \left\langle A\frac{\partial H}{\partial X}, N_i(X)\right\rangle - \left\langle\frac{\partial H}{\partial T}, N_i(X)\right\rangle = 0$$

$$\text{for } i = 1, 2, 3, \ldots, N \quad (24)$$

Hereafter for convenience H^A is written as H. Integration of (24) yields

$$\left\{H\frac{\partial H}{\partial X}\cdot N_i(X)\right\}\Bigg|_{X=0}^{X=1} - \int_0^1 \frac{d}{dX}N_i(X)\cdot\left(H\frac{\partial H}{\partial X}\right)dX$$

$$- A\int_0^1 \frac{\partial H}{\partial X}\cdot N_i(X)\,dX - \int_0^1 \frac{\partial H}{\partial T}\cdot N_i(X)\,dX = 0$$

$$\text{for } i = 1, 2, 3, \ldots, N \quad (25)$$

By substituting the value of H from (21) into (25), a system of N integral equations is obtained as below

$$\sum_{j=1}^{N}\int_0^1 N_i(X)N_j(X)\frac{dZ_j}{dT}\,dX + \frac{1}{2}\sum_{j=1}^{N}\int_0^1 \frac{dN_i(X)}{dX}\frac{dN_j^2(X)}{dX}Z_j^2\,dX$$

$$+ A\sum_{j=1}^{N}\int_0^1 N_i(X)\frac{dN_j(X)}{dX}Z_j\,dX = \left\{H\frac{\partial H}{\partial X}\cdot N_i(X)\right\}\bigg|_{X=1}$$

$$-\left\{H\frac{\partial H}{\partial X}\cdot N_i(X)\right\}\bigg|_{X=0} \quad \text{for } i = 1, 2, 3, \ldots, N \tag{26}$$

or

$$\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)N_j(X)\frac{dZ_j}{dT}\,dX + \frac{1}{2}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e \frac{dN_i(X)}{dX}$$

$$\cdot\frac{dN_j^2(X)}{dX}Z_j^2\,dX + A\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)\frac{dN_j(X)}{dX}Z_j\,dX$$

$$= \left\{H\frac{\partial H}{\partial X}\right\}\bigg|_{X=1} - \left\{H\frac{\partial H}{\partial X}\right\}\bigg|_{X=0} \quad \text{for } i = 1, 2, 3, \ldots, N \tag{27}$$

Eq. (27) can be rewritten

$$[G]\left\{\frac{dZ}{dT}\right\} + [B]\{Z^2\} + A[C]\{Z\} = \{F\} \tag{28}$$

where

$$[G] = G_{ij} = \sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)N_j(X)\,dX \tag{29a}$$

$$[B] = B_{ij} = \frac{1}{2}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e \frac{dN_i(X)}{dX}\frac{dN_j^2(X)}{dX}\,dX \tag{29b}$$

$$[C] = C_{ij} = \sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)\frac{dN_j(X)}{dX}\,dX \tag{29c}$$

$$\{F_i\} = 0 \quad \text{for } i = 2, 3, 4, \ldots, N-1 \tag{29d}$$

$$\{F_1\} = -\left(H\frac{\partial H}{\partial X}\right)\bigg|_{X=0};\quad \{F_N\} = \left(H\frac{\partial H}{\partial X}\right)\bigg|_{X=1} \tag{29e,f}$$

Coefficient matrices were evaluated and are given in the Appendix.

Eq. (28) may be written in finite-difference form

$$[G]\left\{\frac{Z(T+\Delta T) - Z(T)}{\Delta T}\right\} + [B]\{Z^2(T+\Delta T)\}$$

$$+ A[C]\{Z(T+\Delta T)\} = \{F(T)\} \tag{30}$$

Let $Z(T + \Delta T) = Z(T) + V(T)$. Substitution of this relationship in (30) yields

$$[G]\left\{\frac{V(T)}{\Delta T}\right\} + [B]\{Z^2(T) + 2Z(T)V(T) + V^2(T)\}$$

$$+ A[C]\{Z(T) + V(T)\} = \{F(T)\} \tag{31}$$

Neglecting the terms of $O(V^2(T))$ gives

$$[G]\{V(T)\} + \Delta T[B]\{2Z(T)V(T)\} + \Delta TA[C]\{V(T)\}$$

$$= -\Delta T[B]\{Z^2(T)\} - \Delta TA[C]\{Z(T)\} + \Delta T\{F(T)\} \tag{32}$$

or

$$[[G] + 2\Delta T[B]\{Z(T)\} + \Delta TA[C]]\{V(T)\}$$

$$= -\Delta T[B]\{Z^2(T)\} - \Delta TA[C]\{Z(T)\} + \Delta T\{F(T)\} \tag{33}$$

The solution of this system of algebraic equations provides the values of $V(T)$ at different nodes. This $V(T)$ value at a particular node is added to the value of $Z(T)$ at that node to get the value of $Z(T + \Delta T)$ at that particular node for the next time step. To obtain the solution for the water table profile in horizontal aquifers, A is substituted as zero. The resulting expression will be the finite-element solution for falling water tables between two conventional level drains in a horizontal aquifer.

Hybrid Finite Analytic Solution

Pi and Hjelmfelt (1994) solved an extended Dupuit-Forchheimer equation to describe water table profiles and lateral subsurface streamflow in a sloping aquifer using a hybrid finite analytic method based on the approach of Chen (1988). A local linearized 1D Dupuit-Forchheimer equation was solved analytically in space and discretized in time by a simple difference formula. The resultant system of algebraic equations approximated the overall nonlinear effect because the coefficient of diffusion and the term $(\partial h/\partial x)^2$ were treated as constants only in the local regions. A four-point numerical formula provided stable and sufficiently accurate results with simple calculations and without small time steps. Steady-state profiles of water table and lateral subsurface storm flow obtained from their study compared well with the results of previous investigators.

Using the approach of Chen (1988), the hybrid finite analytic solution of nonlinear nondimensionalized equations (13) or (14), along with initial and boundary conditions defined by (15a) and (15b), was obtained. The procedure for obtaining hybrid finite analytic solutions to describe falling water tables between two conventional level drains is as follows.

If H associated with $(\partial^2H/\partial X^2)$ in (13) is replaced by H_a, the dimensionless average depth of flow, it may be written

$$\frac{\partial^2 H}{\partial X^2} + \frac{1}{H_a}\left(\frac{\partial H}{\partial X}\right)^2 - \frac{A}{H_a}\left(\frac{\partial H}{\partial X}\right) = \frac{1}{H_a}\frac{\partial H}{\partial T} \tag{34}$$

Assuming the terms $1/H_a(\partial H/\partial X)^2$, $1/H_a(\partial H/\partial T)$, and A/H_a as constants denoted by C, E, and $2s_1$, respectively, in a small subregion and performing integration, one gets the following equation:

$$\frac{dH}{dX} - 2s_1H = (E - C)X + F \tag{35}$$

The solution of this first-order ordinary differential equation is

$$H(X) = Ge^{2s_1X} - (E - C)\frac{X}{2s_1} + I \tag{36}$$

Discretization of (36) in space and time yields the following equations:

$$H_{i-1}^{n+1} = Ge^{-2s_1\Delta X} + (E - C)\frac{\Delta X}{2s_1} + I \tag{37a}$$

$$H_{i+1}^{n+1} = Ge^{2s_1\Delta X} - (E - C)\frac{\Delta X}{2s_1} + I \tag{37b}$$

$$H_i^{n+1} = G + I \tag{37c}$$

because H_i^{n+1} represents the point where $\Delta X = 0$.

Simplification of these equations yield a tridiagonal matrix

$$A_iH_{i-1}^{n+1} + B_iH_i^{n+1} + C_iH_{i+1}^{n+1} = D_iH_i^n + E_i \tag{38}$$

where

$$A_i = -\frac{e^{s_1\Delta X}}{e^{s_1\Delta X} + e^{-s_1\Delta X}};\quad B_i = 1 + \frac{\Delta X}{2H_as_1\Delta T}\tanh(s_1\Delta X) \tag{39a,b}$$

$$C_i = -\frac{e^{-s_1\Delta X}}{e^{s_1\Delta X} + e^{-s_1\Delta X}};\quad D_i = \frac{\Delta X}{2H_as_1\Delta T}\tanh(s_1\Delta X) \tag{39c,d}$$

$$E_i = \frac{\Delta X}{2H_a s_1}\left(\frac{\partial H}{\partial X}\right)^2 \tanh(s_1 \Delta X) \quad (39e)$$

By solving the above tridiagonal matrix, one may obtain the value of H at any space and time. The water table elevations h in a sloping aquifer at any distance and time may be computed by multiplying H with h_0.

It should be possible to obtain water table elevations in a horizontal aquifer by putting $A = (\alpha L/h_0) = 0$ in the above solution but then the expression becomes indeterminate. However, water table elevations in a horizontal aquifer may be obtained by substituting a very small value of α such as 0.00001 in the above solution without affecting its general stability.

EXPERIMENTAL MODEL

Chauhan (1967) conducted an experimental investigation on a Hele-Shaw viscous flow model. The experiment for the horizontal case was conducted at 68°F using Shell Tellus oil 72, which gave a permeability of 2.328 in./min. The sloping case experiments were conducted at 72°F with a permeability of 2.890 in./min. The initial oil profile was parallel to the impermeable layer in all the tests. Midpoint fall of oil for several time intervals was recorded visually above the impermeable layer and was reported for the cases of 0, 4, 6, and 8% slopes of the impermeable barrier.

RESULTS AND DISCUSSION

The Boussinesq equation itself contains a major simplification as it reduces a 2D problem to a 1D problem by assuming vertical energy losses are negligible compared to the horizontal energy losses. All the analytical and numerical solutions of such linearized and nonlinear Boussinesq equations mentioned above are also approximate solutions. But the solutions of the nonlinear Boussinesq equation, which take into account all the terms, are expected to perform better than those obtained for the linearized equations. To determine the relatively better solution among the solutions studied here, the experimental results of midpoint falling water tables obtained from the Hele-Shaw model for nonsloping and sloping conditions as reported by Chauhan (1967) were compared with the values of midpoint decline of water tables obtained from all the analytical and numerical solutions. To compare the experimental results with the midpoint falling water tables computed from various theoretical solutions, the following parameter values were used. The initial water table above the drains or impermeable barrier $h_0 = 12$ in., specific yield $f = 1.0$, spacing between two drains $L = 100$ in., and hydraulic conductivity for nonsloping and sloping cases $K = 2.328$ and 2.890 in./min, respectively. In numerical solutions the dimensionless time increment ΔT and dimensionless space increment ΔX were taken as 0.0001 and 0.01, respectively. The midpoint falling water tables at 5, 25, 50, 100, 150, 200, 250, and 300 min for 0, 4, 6, and 8% slope obtained from various solutions are given in Tables 1–4.

It may be observed from Table 1 that, for a nonsloping condition, for a small time period in the beginning, the values of water table elevations computed from analytical solution I are higher than those obtained from the experimental model. With time, however, the values of the water table elevation computed from analytical solution I decrease rapidly as compared to the experimental data. The water table elevations obtained from analytical solution II are always higher than those computed from analytical solution I and, for most of the time period, values computed from this solution are higher than those obtained from the experimental data. The values of water table elevations computed by employing finite-difference and finite-element solutions are almost similar and marginally higher than the values obtained from the experimental model. The hybrid finite analytic solution predicts marginally higher values compared to the experimental data, but these values are marginally lower than the values predicted by finite-difference and finite-element solutions. Thus, values of midpoint fall of water table with time as obtained from the hybrid finite analytic solution are closest to the experimental results. The difference in the results obtained from mathematical and experimental solutions may be attributed to the inaccuracy that Boussinesq introduces under the given geometry, including not taking the effect of seepage zone into account.

Note from Tables 2–4 that, for sloping cases (4, 6, and 8% slope), the values of water table elevations computed from various theoretical solutions show an almost similar trend as shown in the nonsloping case and the results computed from

TABLE 1. Experimental and Computed Midpoint Transient Falling Water Tables (in.) for Nonsloping Case

Time (min)	Experimental results	Analytical solution I with Baumann's linearization	Analytical solution II with Werner's linearization	Finite-difference solution	Finite-element solution	Hybrid finite analytic solution
5	11.78	11.993	11.999	11.972	11.972	11.966
25	10.00	10.596	11.276	10.051	10.051	9.985
50	7.80	7.660	9.587	7.984	7.983	7.903
100	5.35	3.853	6.796	5.644	5.643	5.560
150	4.05	1.938	4.815	4.365	4.364	4.284
200	3.25	0.978	3.411	3.558	3.558	3.479
250	2.80	0.496	2.417	3.003	3.003	2.925
300	2.42	0.254	1.712	2.598	2.598	2.521

TABLE 2. Experimental and Computed Midpoint Transient Falling Water Tables for 4% Slope

Time (min)	Experimental results	Analytical solution I with Baumann's linearization	Analytical solution II with Werner's linearization	Finite-difference solution	Finite-element solution	Hybrid finite analytic solution
5	11.75	11.995	11.998	11.926	11.927	11.901
25	9.75	9.830	10.861	9.456	9.456	9.564
50	7.38	6.447	8.794	7.236	7.235	7.242
100	4.86	2.718	5.707	4.902	4.902	4.814
150	3.65	1.148	3.703	3.690	3.690	3.607
200	2.86	0.488	2.402	2.944	2.944	2.864
250	2.47	0.210	1.558	2.437	2.436	2.359
300	1.98	0.093	1.011	2.067	2.067	1.991

TABLE 3. Experimental and Computed Midpoint Transient Falling Water Tables for 6% Slope

Time (min)	Experimental results	Analytical solution I with Baumann's linearization	Anaytical solution II with Werner's linearization	Finite-difference solution	Finite-element solution	Hybrid finite analytic solution
5	11.74	11.994	11.998	11.926	11.927	11.901
25	9.64	9.801	10.845	9.444	9.443	9.452
50	7.20	6.391	8.756	7.208	7.207	7.114
100	4.65	2.662	5.648	4.850	4.849	4.762
150	3.55	1.111	3.642	3.616	3.615	3.533
200	2.62	0.467	2.349	2.849	2.849	2.770
250	2.11	0.199	1.515	2.321	2.321	2.244
300	1.72	0.087	0.977	1.930	1.930	1.855

TABLE 4. Experimental and Computed Midpoint Transient Falling Water Tables for 8% Slope

Time (min)	Experimental results	Analytical solution I with Baumann's linearization	Analytical solution II with Werner's linearization	Finite-difference solution	Finite-element solution	Hybrid finite analytic solution
5	11.72	11.994	11.998	11.926	11.926	11.901
25	9.54	9.762	10.823	9.426	9.426	9.435
50	7.10	6.313	8.703	7.168	7.168	7.075
100	4.57	2.586	5.566	4.775	4.774	4.687
150	3.25	1.061	3.559	3.511	3.510	3.428
200	2.46	0.438	2.276	2.714	2.713	2.634
250	1.91	0.184	1.456	2.154	2.153	2.078
300	1.48	0.080	0.931	1.730	1.729	1.656

TABLE 5. Comparison of Theoretical Solutions with Experimental Solution for Nonsloping and Sloping Cases with $L2$ and Tchebycheff Norms

Norms	Analytical solution I with Baumann's linearization	Analytical solution II with Werner's linearization	Finite-difference solution	Finite-element solution	Hybrid finite analytic solution
		(a) Nonsloping Case (0% Slope)			
$L2$	1.827	1.028	0.251	0.250	0.178
Tchebycheff	2.304	1.787	0.315	0.314	0.234
		(b) Sloping Case (4% Slope)			
$L2$	2.018	0.854	0.114	0.113	0.093
Tchebycheff	2.502	1.414	0.294	0.294	0.186
		(c) Sloping Case (6% Slope)			
$L2$	1.843	0.843	0.178	0.178	0.123
Tchebycheff	2.439	1.556	0.229	0.229	0.188
		(d) Sloping Case (8% Slope)			
$L2$	1.711	0.840	0.219	0.218	0.149
Tchebycheff	2.189	1.603	0.261	0.260	0.181

the hybrid finite analytic solution are closest to the experimental results, followed by the results obtained from finite-element and finite-difference solutions.

To compare the performance of various analytical and numerical solutions with the experimental data, Euclidean $L2$ and Tchebycheff L_∞ norms as reported by Prenter (1975) were employed to measure the goodness and accuracy of various theoretical solutions. The $L2$ norm, which gives the average distance of the theoretical solutions from the experimental model, and the L_∞ norm, which describes the maximum difference between the theoretical solutions and the experimental model, were computed. The values of $L2$ and L_∞ norms are given in Table 5 for nonsloping and sloping conditions, respectively.

Note from Table 5 that for the nonsloping as well as sloping cases the least value of $L2$ and L_∞ norms are obtained for the hybrid finite analytic solution. The observed performances of various methods in decreasing order were finite-element solution, finite-difference solution, analytical solution II with Werner's transformation, and analytical solution I with Baumann's transformation. Thus, for both nonsloping and sloping cases, the hybrid finite analytic solution may be used and considered as the best solution among all the solutions presented here.

All three numerical solutions tested for the given geometry were found convergent and stable at the selected space and time increments, and no significant improvement in results was observed on finer discretization of space and time.

CONCLUSIONS

Analytical solutions were obtained for the linearized Boussinesq equation using Baumann's method and Werner's method of linearization. Numerical solutions using finite-difference and finite-element methods were obtained for the nonlinear form of the Boussinesq equation. The hybrid finite analytic method was also used to solve the nonlinear Boussinesq equation. In the hybrid finite analytic method, the overall nonlinear effect could be preserved by the assembly of locally analytic solutions. Performance of various theoretical solutions with respect to experimental results were studied by comparing the values of midpoint falling water tables obtained from an experimental model with those obtained from theoretical solutions employing $L2$ and L_∞ norms. Results reveal that the hybrid finite analytic solution predicts the best values of the midpoint falling water tables between two drains in a horizontal/sloping aquifer, followed in decreasing order of performance by finite-element, finite-difference, and analytical solutions. However, for all practical purposes, any numerical solution may be used to compute falling water tables and these solutions should be preferred over approximate analytical solutions for the given geometry.

APPENDIX. EVALUATION OF COEFFICIENT MATRICES

$$G_{11} = \frac{1}{3}(X_2 - X_1); \quad G_{NN} = \frac{1}{3}(X_N - X_{N-1}) \quad (40a,b)$$

$$G_{ii} = \frac{1}{3}(X_{i+1} - X_{i-1}) \quad \text{for } i = 2, 3, 4, \ldots, N-1 \quad (40c)$$

$$G_{ii-1} = \frac{1}{6}(X_i - X_{i-1}) \quad \text{for } i = 2, 3, 4, \ldots, N \quad (40d)$$

$$G_{ii+1} = \frac{1}{6}(X_{i+1} - X_i) \quad \text{for } i = 1, 2, 3, \ldots, N-1 \quad (40e)$$

$$B_{11} = \frac{1}{2(X_2 - X_1)}; \quad B_{NN} = \frac{1}{2(X_N - X_{N-1})} \quad (40f,g)$$

$$B_{ii} = \frac{1}{2(X_i - X_{i-1})} + \frac{1}{2(X_{i+1} - X_i)} \quad \text{for } i = 2, 3, 4, \ldots, N-1 \quad (40h)$$

$$B_{ii+1} = -\frac{1}{2(X_{i+1} - X_i)} \quad \text{for } i = 1, 2, 3, \ldots, N-1 \quad (40i)$$

$$B_{ii-1} = \frac{1}{2(X_i - X_{i-1})} \quad \text{for } i = 2, 3, 4, \ldots, N \quad (40j)$$

$$C_{11} = -\frac{1}{2}; \quad C_{NN} = \frac{1}{2}; \quad C_{ii} = 0 \quad (40k\text{–}m)$$

$$C_{ii-1} = -\frac{1}{2}; \quad C_{ii+1} = \frac{1}{2} \quad (40n,o)$$

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