Interaction of Stream and Sloping Aquifer Receiving Constant Recharge

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Abstract: An analytical solution and finite-element numerical solution of a linearized and nonlinear Boussinesq equation, respectively, were obtained to describe water table variation in a semi-infinite sloping/horizontal aquifer caused by the sudden rise or fall of the water level in the adjoining stream. Transient water table profiles in recharging and discharging aquifers having 0, 5, and 10% slopes and receiving zero or constant replenishment from the land surface were computed for $t = 1$ and 5 days by employing analytical and finite-element numerical solutions. The effect of linearization of the nonlinear governing equation, recharge, and slope of the impermeable barrier on water table variation in a semi-infinite flow region was illustrated with the help of a numerical example. Results suggest that linearization of the nonlinear equation has only a marginal impact on the predicted water table heights (with or without considering constant replenishment). The relative errors between the analytical and finite-element numerical solution varied in the range of −0.39 to 1.59%. An increase in slope of the impermeable barrier causes an increase in the water table height at all the horizontal locations, except at the boundaries for the recharging case and a decrease for the discharging case.

INTRODUCTION

Boussinesq (1904) derived a partial differential equation using the principle of continuity and adopting the classical Dupuit Forchheimer assumptions (all streamlines are horizontal) to describe groundwater flow in an unconfined gently sloping aquifer above an impermeable barrier. Werner (1953, 1957) studied the problems of nonartesian aquifers with reference to unsteady flow due to recharge from the ground surface. He used the Boussinesq equation after incorporating the term of recharge and expressed the equation

$$h \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 - \alpha \left(\frac{\partial h}{\partial x}\right) + \frac{R}{K} = \frac{f}{K}\frac{\partial h}{\partial t} \qquad (1a)$$

where h = height of the phreatic surface above the sloping impermeable barrier (L); α = slope of the impermeable barrier; x = space coordinate along horizontal reference axis (L); t = time (T); K = hydraulic conductivity of the aquifer (LT^{-1}); f = drainable porosity (dimensionless); and R = surface applied replenishment, which is equal to R'/f, where R' denotes the speed of replenishment to the water table (i.e., rate of recharge or draft within the soil) (Maasland 1959). The linearized form of (1a), which is obtained by neglecting the term $(\partial h/\partial x)^2$ and replacing the term h associated with $(\partial^2 h/\partial x^2)$ with D, the average depth of flow, has been adopted by a number of researchers and may be written

$$\frac{\partial^2 h}{\partial x^2} - 2s\left(\frac{\partial h}{\partial x}\right) + \frac{R}{KD} = \frac{1}{a}\frac{\partial h}{\partial t} \qquad (1b)$$

where $s = \alpha/2D$ and $a = KD/f$.

Water table variation in a sloping aquifer receiving constant replenishment and interacting with a stream having abrupt rise or fall of water level (as shown in Figs. 1 and 2 for recharging and discharging aquifers, respectively) can be represented mathematically by the nonlinear differential equation [(1a)] or the linearized differential equation [(1b)]. The initial and boundary conditions corresponding to (1a) and (1b) may be written

$$h = h_1; \quad x = 0; \quad t > 0 \qquad (2)$$

$$h = h_0; \quad x > 0; \quad t = 0 \qquad (3)$$

$$h = h_0; \quad x \to \infty; \quad t > 0 \qquad (4)$$

where h_1 and h_0 denote water levels in the stream at $x = 0$ and in the aquifer at $x = \infty$. The definition sketches of the water table profile in recharging and discharging aquifers are given in Figs. 1 and 2, respectively.

Many investigators have studied the water table variation in a semi-infinite horizontal aquifer, resulting from the sudden rise or drop of the water table in the adjoining stream, using analytical and numerical approaches. Such studies include those by Edelman (1947), Polubarinova-Kochina (1948, 1949), Verigin (1949), Hornberger et al. (1970), Zucker et al. (1973), Marino (1973), Sidiropoulos et al. (1984), Tolikas et al. (1984), Lockington (1997), Workman et al. (1997), Serrano and Workman (1998), Upadhyaya and Chauhan (1998), and Upadhyaya (1999). Only a few studies seem to be related to stream and sloping aquifer interaction. Polubarinova-Kochina (1962) obtained an analytical solution of the linearized Boussinesq equation to describe seepage from one canal to another on sloping bedrock. Yussuff et al. (1994) obtained a finite difference numerical solution of the nonlinear Boussinesq equation characterizing the phreatic surface in a semi-infinite sloping aquifer. They also obtained an analytical solution by modifying Polubarinova-Kochina's solution (1962) of a generalized boundary condition to describe seepage from a canal in a semi-infinite flow region. They observed that phreatic sur-

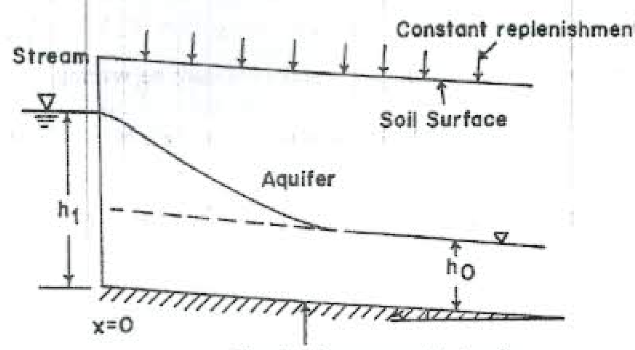

FIG. 1. Definition Sketch for Recharging Aquifer with Constant Replenishment

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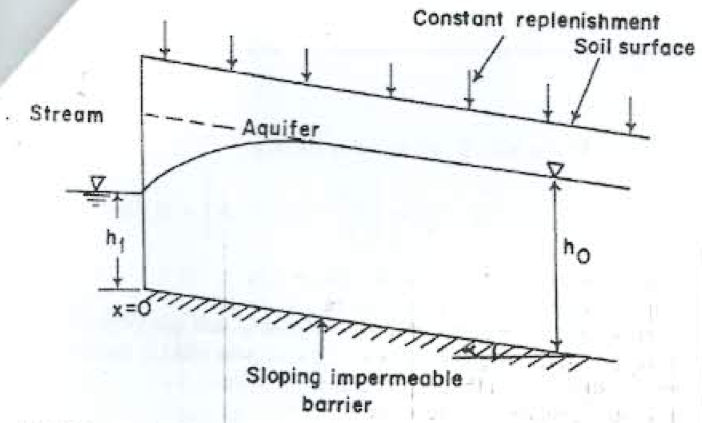

FIG. 2. Definition Sketch for Discharging Aquifer with Constant Replenishment

face values predicted by the numerical solution were overall higher for all distances and times than those predicted by the analytical solution of the linearized Boussinesq equation. No studies were found in the literature to describe the variation of the water table in a sloping semi-infinite aquifer receiving constant replenishment and interacting with a stream having a sudden rise or fall of water level. The objective of this study is to obtain analytical and finite-element numerical solutions to predict a water table profile due to stream and sloping aquifer interaction with constant replenishment from the land surface.

ANALYTICAL SOLUTION

An analytical solution to the linearized Boussinesq equation [(1*b*)], incorporating constant replenishment with initial and boundary conditions [(2)–(4)], was obtained by devising the transformation to convert (1*b*) into a heat flow equation. The transformation is

$$h_c = (h - h_0) = v e^{sx - s^2 at} + \frac{R_0 t}{f} \tag{5}$$

where v = new transformed variable; and R_0 = constant surface applied replenishment.

With this transformation, the boundary-value problem becomes

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{a}\frac{\partial v}{\partial t} \tag{6}$$

$$v(x, 0) = 0 \quad \text{at } t = 0 \quad \text{for } x > 0 \tag{7a}$$

$$v(0, t) = \left[h_1 - h_0 - \frac{R_0 t}{f}\right] e^{s^2 at} = f(t) \quad \text{at } t > 0 \quad \text{for } x = 0 \tag{7b}$$

$$v(x, t) = 0 \quad \text{at } t > 0 \quad \text{for } x \to 0 \tag{7c}$$

Laplace transform of (6), (7*a*), and (7*b*) may be written

$$\frac{d^2\bar{v}(x, p)}{dx^2} - \frac{p}{a}\bar{v}(x, p) = 0 \quad \text{for } 0 < x < \infty \tag{8}$$

$$\bar{v}(x, p) = \left[\frac{(h_1 - h_0)}{(p - s^2 a)} - \frac{(R_0)}{f(p - s^2 a)^2}\right] = \bar{f}(p) \quad \text{at } x = 0 \tag{9a}$$

$$\bar{v}(x, p) = 0 \quad \text{as } x \to \infty \tag{9b}$$

where $\bar{v}$ = Laplace transform of v; and p = Laplace variable.

The generalized solution to this boundary value problem as reported by Ozisik (1980) is

$$\bar{v}(x, p) = \bar{f}(p) \cdot \bar{g}(x, p) \tag{10}$$

where

$$\bar{g}(x, p) = \exp(-x\sqrt{p/a})$$

Substituting the values of $\bar{f}(p)$ and $\bar{g}(x, p)$ in (10) it becomes

$$\bar{v}(x, p) = \frac{(h_1 - h_0)\exp(-x\sqrt{p/a})}{(p - s^2 a)} - \frac{R_0 \exp(-x\sqrt{p/a})}{f(p - s^2 a)^2} \tag{11}$$

Taking the inverse of the Laplace transformation as reported in Carslaw and Jaeger (1959), (11) yields

$$v(x, t) = \left(\frac{h_1 - h_0}{2}\right) e^{s^2 at}\left\{e^{-sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} - s\sqrt{at}\right] + e^{sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} + s\sqrt{at}\right]\right\} - \frac{R_0}{2f} e^{s^2 at}\left\{\left(t - \frac{x}{2as}\right) e^{-sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} - s\sqrt{at}\right] + \left(t + \frac{x}{2as}\right) e^{sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} + s\sqrt{at}\right]\right\} \tag{12}$$

Again applying the inverse of transformation (5), the solution in terms of $h(x, t)$ may be written

$$h(x, t) = \left(\frac{h_1 - h_0}{2}\right)\left\{\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} - s\sqrt{at}\right] + e^{2sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} + s\sqrt{at}\right]\right\} - \frac{R_0}{2f}\left\{\left(t - \frac{x}{2as}\right)\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} - s\sqrt{at}\right] + \left(t + \frac{x}{2as}\right) e^{2sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} + s\sqrt{at}\right]\right\} + \frac{R_0 t}{f} + h_0 \tag{13}$$

SPECIAL CASES

Case 1

If there is no recharge occurring in a sloping aquifer, the solution to describe the water table variation for such a condition as a result of stream aquifer interaction can be obtained by putting $R_0 = 0$ in (13) and written

$$h(x, t) = \left(\frac{h_1 - h_0}{2}\right)\left\{\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} - s\sqrt{at}\right] + e^{2sx}\,\mathrm{erfc}\left[\frac{x}{2\sqrt{at}} + s\sqrt{at}\right]\right\} + h_0 \tag{14}$$

This solution is similar to the one reported by Polubarinova-Kochina (1962) to describe the abrupt rise or fall of the water table as a result of stream and sloping aquifer interaction in a semi-infinite flow region.

Case 2

An analytical solution for the water table variation in the case of a stream and horizontal aquifer interaction with or without constant recharge should be possible to obtain by putting $s = 0$ in (13), but the expression becomes indeterminate when s is substituted as zero in (13). Therefore, the analytical solution for such a flow problem was obtained independently and is presented below.

The transformation used to convert (1*b*) with $s = 0$ into a heat flow equation is

$$h_c = (h - h_0) = v + \frac{R_0 t}{f} \tag{15}$$

With this transformation, the governing partial differential equation is converted to the heat flow equation [(6)] and initial and boundary conditions become

$$v(x, 0) = 0 \quad \text{at } t = 0 \quad \text{for } x > 0 \tag{16a}$$

$$v(0, t) = \left[h_1 - h_0 - \frac{R_0 t}{f} \right] = f(t) \quad \text{at } t > 0 \quad \text{for } x = 0 \tag{16b}$$

$$v(x, t) = 0 \quad \text{at } t > 0 \quad \text{for } x \to \infty \tag{16c}$$

Applying the Laplace transformation to the boundary conditions [(16*b*) and (16*c*)]

$$\bar{v}(x, p) = \left[\frac{(h_1 - h_0)}{p} - \frac{R_0}{fp^2} \right] = \bar{f}(p) \quad \text{at } x = 0 \tag{17a}$$

$$\bar{v}(x, p) = 0 \quad \text{as } x \to \infty \tag{17b}$$

Using the generalized solution to such a boundary-value problem as given by Ozisik (1980), the expression for $\bar{v}(x, p)$ is written

$$\bar{v}(x, p) = \frac{(h_1 - h_0)}{p} \exp(-x\sqrt{p/a}) - \frac{R_0}{fp^2} \exp(-x\sqrt{p/a}) \tag{18}$$

Taking the inverse of the Laplace transformation as reported in Carslaw and Jaeger (1959), (18) gives

$$v(x, t) = (h_1 - h_0)\text{erfc}\left(\frac{x}{2\sqrt{at}}\right) - \frac{R_0}{f}\left[\left(t + \frac{x^2}{2a}\right)\text{erfc}\left(\frac{x}{2\sqrt{at}}\right) - x\left(\frac{t}{a\pi}\right)^{1/2}\exp\left(-\frac{x^2}{4at}\right)\right] \tag{19}$$

Again using the inverse of transformation (15), the solution in terms of $h(x, t)$ may be written

$$h(x, t) = \frac{R_0 t}{f} + h_0 + (h_1 - h_0)\text{erfc}\left(\frac{x}{2\sqrt{at}}\right) - \frac{R_0}{f}\left[\left(t + \frac{x^2}{2a}\right)\text{erfc}\left(\frac{x}{2\sqrt{at}}\right) - x\left(\frac{t}{a\pi}\right)^{1/2}\exp\left(-\frac{x^2}{4at}\right)\right] \tag{20}$$

Case 3

If the water table in a horizontal aquifer changes in response to a sudden change in water level in the adjoining stream and the effect of recharge is neglected, the solution to describe the water table variation in the horizontal aquifer due to such an interaction can be obtained by putting $R_0 = 0$ in (20)

$$h(x, t) = h_0 + (h_1 - h_0)\text{erfc}\left(\frac{x}{2\sqrt{at}}\right) \tag{21}$$

This solution is similar to the one reported by Polubarinova-Kochina (1962), which describes the rise or fall of the water table in a horizontal aquifer caused by a sudden change of water level of an interacting stream.

FINITE-ELEMENT SOLUTION

The finite-element solution of the nonlinear Boussinesq equation, as shown by (1) with initial and boundary conditions as $h(x, 0) = h_0$, $h(0, t) = h_1$, and $h(\infty, t) = h_0$, was obtained using Galerkin's method, for which the details are given in Pinder and Gray (1977). To carry out a finite-element analysis, (1) may be written

$$L(h) = \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) - \alpha\left(\frac{\partial h}{\partial x}\right) + \frac{R_0}{K} - \frac{f}{K}\frac{\partial h}{\partial t} = 0 \tag{22}$$

As in Galerkin's finite-element method, the solution is approximated by $h^A(x, t)$ with the help of the basis functions

$$h^A(x, t) = \sum_{i=1}^{N} z_i(t) N_i(x) \tag{23}$$

in which $N_i(x)$, a linear basis function associated with each node x_i (Prenter 1975) is

$$N_i(x) = \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \quad \text{for } x_{i-1} \le x \le x_i$$

$$N_i(x) = \frac{(x_{i+1} - x)}{(x_{i+1} - x_i)} \quad \text{for } x_i \le x \le x_{i+1}$$

and unknown coefficients $z_i(t)$ are determined by forcing the residual $L(h^A)$ to be orthogonal to the basis functions $N_i(x)$, $i = 1, 2, 3, \ldots, N$ (here $i = 1$ denotes $x = 0$ and $i = N$ denotes $x = \infty$). For this, the inner product of $L(h^A)$ with $N_i(X)$ has to be zero; i.e.

$$\langle L(h^A) \cdot N_i(x) \rangle = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \tag{24}$$

where $\langle \cdot \rangle$ represents the dot product.

Hereafter for convenience h^A is written as h. Substitution of (22) in (24) yields

$$\left\langle \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \cdot N_i(x) \right\rangle - \left\langle \alpha \frac{\partial h}{\partial x} \cdot N_i(x) \right\rangle + \left\langle \left(\frac{R_0}{K}\right) \cdot N_i(x) \right\rangle - \left\langle \frac{f}{K}\frac{\partial h}{\partial t} \cdot N_i(x) \right\rangle = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \tag{25}$$

Because in a semi-infinite flow region x varies from 0 to ∞, integration of (25) from $x = 0$ to $x = \infty$ yields

$$\int_0^\infty \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \cdot N_i(x)\, dx - \alpha \int_0^\infty \frac{\partial h}{\partial x} \cdot N_i(x)\, dx + \frac{R_0}{K}\int_0^\infty N_i(x)\, dx - \frac{f}{K}\int_0^\infty \frac{\partial h}{\partial t} \cdot N_i(x)\, dx = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \tag{26}$$

or

$$\left\{h\frac{\partial h}{\partial x} \cdot N_i(x)\right\}\Bigg|_{x=0}^{x=\infty} - \int_0^\infty \frac{d}{dx}N_i(x) \cdot \left(h\frac{\partial h}{\partial x}\right) dx - \alpha\int_0^\infty \frac{\partial h}{\partial x} \cdot N_i(x)\, dx + \frac{R_0}{K}\int_0^\infty N_i(x)\, dx - \frac{f}{K}\int_0^\infty \frac{\partial h}{\partial t} \cdot N_i(x)\, dx = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \tag{27}$$

Substituting the value of h from (23) into (27), a system of N integral equations is obtained

$$\frac{f}{K}\sum_{j=1}^{N}\int_0^\infty N_i(x)N_j(x)\frac{dz_j}{dt}\, dx + \frac{1}{2}\sum_{j=1}^{N}\int_0^\infty \frac{dN_i(x)}{dx}\frac{dN_j^2(x)}{dx} z_j^2\, dx + \alpha\sum_{j=1}^{N}\int_0^\infty N_i(x)\frac{dN_j(x)}{dx} Z_j\, dx = \left\{h\frac{\partial h}{\partial x} \cdot N_i(x)\right\}\Bigg|_{x=\infty} - \left\{h\frac{\partial h}{\partial x} \cdot N_i(x)\right\}\Bigg|_{x=0} + \frac{R_0}{K}\int_0^\infty N_i(x)\, dx \quad \text{for } i = 1, 2, 3, \ldots, N \tag{28}$$

or

$$\frac{f}{K}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(x)N_j(x)\frac{dz_j}{dt}\, dx + \frac{1}{2}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e \frac{dN_i(x)}{dx}\frac{dN_j^2(x)}{dx} z_j^2\, dx + \alpha\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(x)\frac{dN_j(x)}{dx} z_j\, dx = \left\{h\frac{\partial h}{\partial x}\right\}\Bigg|_{x=\infty} - \left\{h\frac{\partial h}{\partial x}\right\}\Bigg|_{x=0} + \frac{R_0}{K}\sum_{e=1}^{M}\int_e N_i(x)\, dx \quad \text{for } i = 1, 2, 3, \ldots, N \tag{29}$$

Eq. (29) can be rewritten

$$[G]\left\{\frac{dz}{dt}\right\} + [B]\{z^2\} + [C]\{Z\} = \{F\} \quad (30)$$

where

$$[G] = G_{ij} = \frac{f}{K}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(x)N_j(x)\,dx \quad (31a)$$

$$[B] = B_{ij} = \frac{1}{2}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e \frac{dN_i(x)}{dx}\frac{dN_j^2(x)}{dx}\,dx \quad (31b)$$

$$[C] = C_{ij} = \alpha\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(x)\frac{dN_j(x)}{dx}\,dx \quad (31c)$$

$$\{F_i\} = \frac{R_0}{K}\sum_{e=1}^{M}\int_e N_i(x)\,dx \quad \text{for } i = 2, 3, 4, \ldots, N-1 \quad (31d)$$

$$\{F_1\} = \frac{R_0}{K}\sum_{e=1}^{M}\int_e N_1(x)\,dx - \left(h\frac{\partial h}{\partial x}\right)\bigg|_{x=0} \quad (31e)$$

$$\{F_N\} = \frac{R_0}{K}\sum_{e=1}^{M}\int_e N_N(x)\,dx + \left(h\frac{\partial h}{\partial x}\right)\bigg|_{x=\infty} \quad (31f)$$

Coefficient matrices obtained from the above are given in the Appendix.

Eq. (30) may be written in finite difference form

$$[G]\left\{\frac{z(t+\Delta t) - z(t)}{\Delta t}\right\} + [B]\{z^2(t+\Delta t)\}$$

$$++ [C]\{z(t+\Delta t)\} = \{F(t)\} \quad (32)$$

Let $z(t + \Delta t) = z(t) + v(t)$. Substituting this relationship in (33) yields

$$[G]\left\{\frac{v(t)}{\Delta t}\right\} + [B]\{z^2(t) + 2z(t)v(t) + v^2(t)\}$$

$$+ [C]\{z(t) + v(t)\} = \{F(t)\} \quad (33)$$

Neglecting the term $[v^2(t)]$, which is the square of the difference of values of z at $t + \Delta t$ and t time steps, gives

$$[[G] + \Delta t[B]\{2z(t)\} + \Delta t[C]]\{v(t)\} = -\Delta t[B]\{z^2(t)\}$$

$$- \Delta t[C]\{z(t)\} + \Delta t\{F(t)\} \quad (34)$$

The solution of this system of algebraic equations provides the values of $v(t)$ at different nodes.

The $z(t)$ values in the flow domain at $t = 0$ are known from the initial condition. The value of $z(t + \Delta t)$ at a particular node and time step $(t + \Delta t)$ can be obtained by adding the $v(t)$ value at a particular node to the value of $z(t)$ at that node. In this way, using the known values of z at $t = 0$, computing v at $t = 0$ from (34), and adding the values, one gets the value of z at the next time step $t + \Delta t$. Repeating this process one can get the values of z at any time. To obtain the solution for the water table profile in horizontal aquifers, the term α in $[C]$ is substituted as zero.

RESULTS AND DISCUSSION

The analytical and finite-element numerical solutions describing the variation of the spatial and temporal distribution of the water table in a sloping and horizontal aquifer caused by an abrupt rise or fall of the water level in the adjoining stream or canal, as obtained above, were evaluated by considering a numerical example. The effect of constant replenishment from the land surface on the water table in recharging and discharging aquifers interacting with a stream in a semi-infinite flow region was also evaluated. A numerical example, which was considered for the purpose of comparison of results obtained from these solutions, is given below.

Numerical Example

Assume the flow of water in a shallow sand aquifer with hydraulic conductivity $K = 20$ m/day and specific yield $f = 0.27$. The aquifer was considered to be underlain by an impermeable barrier having 0, 5, and 10% slopes and initially having a uniform water level elevation $h_0 = 2$ m. The water level in the adjoining trench was instantaneously raised to the elevation $h_1 = 3$ m to provide an interacting recharging aquifer. Similarly, when the water level in the aquifer was at an elevation of $h_0 = 3$ m, and in the canal/trench it was at an elevation of $h_1 = 2$ m, it provided an interacting discharging aquifer. A constant replenishment of 5 mm/day was assumed. In numerical solutions, the values of time increment Δt and space increment Δx were considered as 0.0025 days and 2 m, respectively. The resulting water table profiles in the aquifer for both cases from $t = 1$ to 5 days were determined by both the solutions; however, only the results for 1 and 5 days have been discussed below.

Water Table Variation in Horizontal/Sloping Aquifer due to Stream Aquifer Interaction as Obtained from Analytical Solution and Finite-Element Numerical Solution

Water table variation in the recharging and discharging sloping and horizontal aquifer (which is receiving constant or no replenishment from the land surface) interacting with a stream/canal having a sudden rise or fall of water level, was computed employing the analytical solution and the finite-element nu-

TABLE 1. Comparison of Water Table Heights for $t = 1$ Day as Predicted by Analytical Solution and Finite-Element Numerical Solution for Horizontal and Sloping Recharging Aquifer without Replenishment

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	3.000	3.000	3.000	3.000	3.000	3.000
10.0	2.603	2.638	2.663	2.694	2.719	2.746
20.0	2.299	2.318	2.361	2.387	2.428	2.458
30.0	2.119	2.116	2.159	2.161	2.206	2.215
40.0	2.038	2.030	2.055	2.048	2.079	2.073
50.0	2.009	2.005	2.015	2.010	2.025	2.018
60.0	2.002	2.001	2.004	2.002	2.006	2.003
70.0	2.000	2.000	2.001	2.000	2.001	2.000
80.0	2.000	2.000	2.000	2.000	2.000	2.000

TABLE 2. Comparison of Water Table Heights for $t = 1$ Day as Predicted by Analytical Solution and Finite-Element Numerical Solution for Horizontal and Sloping Recharging Aquifer with Constant Replenishment at 5 mm/day

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	3.000	3.000	3.000	3.000	3.000	3.000
10.0	2.614	2.648	2.673	2.704	2.729	2.755
20.0	2.315	2.334	2.377	2.402	2.443	2.473
30.0	2.137	2.134	2.176	2.179	2.224	2.233
40.0	2.056	2.048	2.074	2.066	2.097	2.092
50.0	2.028	2.024	2.034	2.029	2.043	2.036
60.0	2.020	2.019	2.022	2.020	2.024	2.022
70.0	2.019	2.019	2.019	2.019	2.020	2.019
80.0	2.019	2.019	2.019	2.019	2.019	2.019

merical solution. Water table elevations in recharging and discharging aquifers at t = 1 and 5 days are given in Tables 1–9 and presented below separately for recharging and discharging aquifers.

Recharging Aquifer

Water table elevations in a recharging aquifer having 0, 5, and 10% slopes, receiving zero or constant replenishment, are given in Tables 1 and 2 at t = 1 day and in Tables 3 and 4 at t = 5 days. It may be observed from these tables that water table heights predicted by the analytical solution are lower than those obtained from the finite-element numerical solution near the stream/canal interface, and at certain distances away from the canal, the analytical solution predicts marginally higher values than the numerical solution. Finally, water table elevations predicted by both methods become constant and parallel to the impermeable barrier. The difference in water table elevations may be attributed to the linearization of the nonlinear Boussinesq equation in the analytical approach. It

TABLE 3. Comparison of Water Table Heights for t = 5 Days as Predicted by Analytical Solution and Finite-Element Numerical Solution for Horizontal and Sloping Recharging Aquifer without Replenishment

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	3.000	3.000	3.000	3.000	3.000	3.000
10.0	2.816	2.838	2.888	2.901	2.940	2.944
20.0	2.642	2.675	2.763	2.786	2.860	2.872
30.0	2.486	2.518	2.631	2.662	2.763	2.783
40.0	2.353	2.377	2.502	2.534	2.654	2.682
50.0	2.245	2.258	2.384	2.410	2.541	2.571
60.0	2.163	2.165	2.281	2.297	2.430	2.458
70.0	2.104	2.098	2.196	2.203	2.328	2.349
80.0	2.063	2.055	2.131	2.129	2.239	2.251
90.0	2.036	2.029	2.084	2.077	2.172	2.170
100.0	2.020	2.014	2.051	2.043	2.114	2.107
110.0	2.011	2.006	2.031	2.022	2.072	2.064
120.0	2.005	2.003	2.017	2.011	2.044	2.035
130.0	2.003	2.001	2.009	2.005	2.027	2.018
140.0	2.001	2.000	2.004	2.002	2.015	2.009
150.0	2.001	2.000	2.002	2.001	2.008	2.004
160.0	2.000	2.000	2.001	2.000	2.007	2.002
170.0	2.000	2.000	2.001	2.000	2.006	2.001
180.0	2.000	2.000	2.000	2.000	2.000	2.000

TABLE 4. Comparison of Water Table Heights for t = 5 Days as Predicted by Analytical Solution and Finite-Element Numerical Solution for Horizontal and Sloping Recharging Aquifer with Constant Replenishment at 5 mm/day

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	3.000	3.000	3.000	3.000	3.000	3.000
10.0	2.846	2.864	2.912	2.921	2.959	2.961
20.0	2.693	2.722	2.806	2.825	2.895	2.904
30.0	2.552	2.582	2.689	2.716	2.812	2.828
40.0	2.429	2.453	2.571	2.600	2.715	2.738
50.0	2.328	2.342	2.461	2.486	2.611	2.638
60.0	2.250	2.254	2.364	2.381	2.507	2.533
70.0	2.193	2.191	2.283	2.291	2.410	2.431
80.0	2.154	2.148	2.220	2.221	2.326	2.338
90.0	2.128	2.122	2.174	2.170	2.259	2.261
100.0	2.112	2.107	2.142	2.136	2.204	2.200
110.0	2.103	2.099	2.122	2.116	2.163	2.157
120.0	2.098	2.096	2.109	2.104	2.136	2.129
130.0	2.095	2.094	2.101	2.098	2.118	2.112
140.0	2.094	2.093	2.097	2.095	2.107	2.102
150.0	2.093	2.093	2.095	2.094	2.100	2.097
160.0	2.093	2.093	2.094	2.093	2.098	2.094
170.0	2.093	2.093	2.093	2.093	2.097	2.093
180.0	2.093	2.093	2.093	2.093	2.093	2.093

TABLE 5. Comparison of Water Table Heights for t = 1 Day as Predicted by Analytical and Finite-Element Numerical Solution for Horizontal/Sloping Discharging Aquifer without Replenishment

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	2.000	2.000	2.000	2.000	2.000	2.000
10.0	2.397	2.432	2.337	2.371	2.281	2.312
20.0	2.701	2.717	2.639	2.660	2.572	2.598
30.0	2.881	2.878	2.841	2.842	2.794	2.799
40.0	2.962	2.955	2.945	2.938	2.921	2.915
50.0	2.991	2.986	2.985	2.980	2.975	2.970
60.0	2.998	2.997	2.996	2.994	2.994	2.991
70.0	3.000	2.999	2.999	2.999	2.999	2.998
80.0	3.000	3.000	3.000	3.000	3.000	3.000
90.0	3.000	3.000	3.000	3.000	3.000	3.000

TABLE 6. Comparison of Water Table Heights for t = 1 Day as Predicted by Analytical and Finite-Element Numerical Solution for Horizontal/Sloping Discharging Aquifer with Constant Replenishment at 5 mm/day

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	2.000	2.000	2.000	2.000	2.000	2.000
10.0	2.408	2.444	2.348	2.383	2.290	2.322
20.0	2.717	2.733	2.654	2.675	2.587	2.613
30.0	2.899	2.895	2.859	2.859	2.811	2.816
40.0	2.981	2.973	2.963	2.956	2.939	2.933
50.0	3.009	3.005	3.003	2.998	2.993	2.988
60.0	3.017	3.015	3.015	3.013	3.012	3.009
70.0	3.018	3.018	3.018	3.017	3.017	3.016
80.0	3.018	3.018	3.018	3.018	3.018	3.018
90.0	3.018	3.018	3.018	3.018	3.019	3.018

TABLE 7. Comparison of Water Table Heights for t = 5 Days as Predicted by Analytical and Finite-Element Numerical Solution for Horizontal/Sloping Discharging Aquifer without Replenishment

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	2.000	2.000	2.000	2.000	2.000	2.000
10.0	2.184	2.212	2.112	2.128	2.060	2.066
20.0	2.358	2.394	2.237	2.266	2.140	2.156
30.0	2.514	2.546	2.369	2.402	2.237	2.262
40.0	2.647	2.669	2.498	2.529	2.346	2.376
50.0	2.755	2.765	2.616	2.640	2.459	2.490
60.0	2.837	2.838	2.719	2.733	2.570	2.596
70.0	2.896	2.892	2.804	2.809	2.672	2.690
80.0	2.937	2.931	2.869	2.867	2.761	2.770
90.0	2.964	2.957	2.916	2.911	2.828	2.835
100.0	2.980	2.974	2.949	2.942	2.886	2.886
110.0	2.989	2.985	2.969	2.964	2.928	2.923
120.0	2.995	2.992	2.983	2.978	2.956	2.950
130.0	2.997	2.996	2.991	2.988	2.973	2.969
140.0	2.999	2.998	2.996	2.993	2.985	2.981
150.0	2.999	2.999	2.998	2.996	2.992	2.989
160.0	3.000	2.999	2.999	2.998	2.993	2.994
170.0	3.000	3.000	2.999	2.999	2.994	2.997
180.0	3.000	3.000	3.000	3.000	3.000	2.998
190.0	3.000	3.000	3.000	3.000	3.000	2.999
200.0	3.000	3.000	3.000	3.000	3.000	3.000

TABLE 8. Comparison of Water Table Heights for $t = 5$ Days as Predicted by Analytical and Finite-Element Numerical Solution for Horizontal/Sloping Discharging Aquifer with Constant Replenishment at 5 mm/day

	0% Slope		5% Slope		10% Slope	
X (m)	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution	Analytical solution	Finite-element solution
0.0	2.000	2.000	2.000	2.000	2.000	2.000
10.0	2.213	2.246	2.135	2.156	2.079	2.088
20.0	2.409	2.448	2.280	2.313	2.176	2.195
30.0	2.580	2.613	2.427	2.464	2.286	2.316
40.0	2.724	2.744	2.567	2.600	2.407	2.441
50.0	2.838	2.846	2.694	2.717	2.529	2.562
60.0	2.924	2.923	2.802	2.815	2.647	2.674
70.0	2.986	2.979	2.890	2.894	2.755	2.772
80.0	3.028	3.019	2.958	2.954	2.847	2.855
90.0	3.055	3.047	3.007	3.000	2.915	2.922
100.0	3.072	3.065	3.041	3.032	2.975	2.974
110.0	3.082	3.077	3.060	3.055	3.019	3.013
120.0	3.087	3.084	3.075	3.070	3.047	3.041
130.0	3.090	3.088	3.083	3.079	3.065	3.060
140.0	3.091	3.090	3.088	3.085	3.077	3.073
150.0	3.092	3.091	3.090	3.089	3.084	3.081
160.0	3.092	3.092	3.091	3.090	3.084	3.086
170.0	3.092	3.092	3.092	3.092	3.084	3.089
180.0	3.093	3.092	3.092	3.092	3.092	3.091
190.0	3.093	3.092	3.092	3.092	3.092	3.092
200.0	3.093	3.092	3.092	3.092	3.092	3.092

TABLE 9. Range of Relative Percentage Difference of Analytical Solution with Respect to Numerical Solution

	Recharging Aquifer		Discharging Aquifer	
Slope of aquifer (%)	Without recharge	With constant recharge	Without recharge	With constant recharge
0	−0.39–1.33	−0.39–1.28	−0.24–1.50	−0.30–1.59
5	−0.45–1.27	−0.39–1.22	−0.24–1.43	−0.30–1.50
10	−0.45–1.22	−0.34–1.21	−0.24–1.34	−0.23–1.41

may also be seen from Tables 1–4 that with an increase in slope of the impermeable barrier the water table height corresponding to a particular space coordinate (except boundaries) as obtained from both the solutions increases. Comparison of water table heights at $t = 1$ and 5 days (without replenishment in Tables 1 and 3 and with constant replenishment in Tables 2 and 4) shows that with increase in time the water table elevation at a particular space coordinate increases. At $t = 5$ days the water table profile becomes parallel to the impermeable barrier at larger distances compared to $t = 1$ day. The effect of a constant replenishment of 5 mm/day in the recharging aquifer at $t = 1$ and 5 days can be observed because the values of the water table elevation corresponding to a particular space coordinate as obtained from both the solutions are larger in Tables 2 and 4 than in Tables 1 and 3.

Discharging Aquifer

Water table elevations in a discharging aquifer having 0, 5, and 10% slopes, receiving zero or constant replenishment, are given in Tables 5 and 6 at $t = 1$ day and in Tables 7 and 8 at $t = 5$ days. It may be observed from these tables that, compared to the finite-element numerical solution, the analytical solution underestimates the water table heights near the stream interface and marginally overestimates them with increasing distance after a certain point. Finally, the water table elevations predicted by both methods become constant and parallel to the impermeable barrier. It may be seen from Tables 5–8 that, contrary to a recharging aquifer, with an increasing slope of the impermeable barrier, the water table height corresponding to a specific space coordinate (except boundaries) as obtained from both methods decreases. Comparison of water table heights at $t = 1$ and 5 days (without replenishment in Tables 5 and 7 and with constant replenishment in Tables 6 and 8) shows that with an increase in time the water table elevation at a specific space coordinate decreases. At $t = 5$ days the water table profile becomes parallel to the impermeable barrier at larger distances from the stream. The effect of constant replenishment of 5 mm/day in the discharging aquifer at $t = 1$ and 5 days can be observed because the values of water table elevation corresponding to a particular space coordinate as obtained from both the solutions are larger in Tables 6 and 8 than in Tables 5 and 7.

CONCLUSIONS

An analytical solution of the linearized Boussinesq equation and a finite-element numerical solution of the nonlinear Boussinesq equation were obtained to describe water table variation in a sloping/horizontal aquifer receiving constant replenishment and interacting with the adjoining stream having an abrupt rise or drop of water level. Compared to the finite-element numerical solution, the analytical solution underestimates the water table elevations in the aquifer up to a certain distance and thereafter it marginally overestimates. Finally, the water table profiles obtained from both methods attain a constant value and become parallel to the impermeable barrier. Both methods show that, in the recharging aquifer, the water table is observed to be consistently higher at $t = 5$ days than at $t = 1$ day, whereas, in the discharging aquifer, the water table is observed to be consistently higher at $t = 1$ day than at $t = 5$ days. Because of the effect of constant replenishment at 5 mm/day, water tables are observed to be higher compared to zero replenishment at both $t = 1$ day and $t = 5$ days. For the example considered here, the maximum relative difference in values of water table heights predicted by the analytical and finite-element numerical method varies in the range of −0.39 to 1.59%. It is suggested, however, that finite-element numerical solution of the nonlinear Boussinesq equation should be used because of its accuracy and easy computation compared to the analytical solution of the linearized Boussinesq equation.

APPENDIX. COEFFICIENT MATRICES

$$G_{11} = \frac{f}{3K}(x_2 - x_1) \tag{35a}$$

$$G_{NN} = \frac{f}{3K}(x_N - x_{N-1}) \tag{35b}$$

$$G_{ii} = \frac{f}{3K}(x_{i+1} - x_{i-1}) \quad \text{for } i = 2, 3, 4, \ldots, N-1 \tag{35c}$$

$$G_{ii-1} = \frac{f}{6K}(x_i - x_{i-1}) \quad \text{for } i = 2, 3, 4, \ldots, N \tag{35d}$$

$$G_{ii+1} = \frac{f}{6K}(x_{i+1} - x_i) \quad \text{for } i = 1, 2, 3, \ldots, N-1 \tag{35e}$$

$$B_{11} = \frac{1}{2(x_2 - x_1)} \tag{35f}$$

$$B_{NN} = \frac{1}{2(x_N - x_{N-1})} \tag{35g}$$

$$B_{ii} = \frac{1}{2(x_i - x_{i-1})} + \frac{1}{2(x_{i+1} - x_i)} \quad \text{for } i = 2, 3, 4, \ldots, N-1 \tag{35h}$$

$$B_{ii+1} = -\frac{1}{2(x_{i+1} - x_i)} \quad \text{for } i = 1, 2, 3, \ldots, N-1 \qquad (35i)$$

$$B_{ii-1} = -\frac{1}{2(x_i - x_{i-1})} \quad \text{for } i = 2, 3, 4, \ldots, N \qquad (35j)$$

$$C_{11} = -\frac{\alpha}{2} \qquad (35k)$$

$$C_{NN} = \frac{\alpha}{2} \qquad (35l)$$

$$C_{ii} = 0 \qquad (35m)$$

$$C_{ii-1} = -\frac{\alpha}{2} \qquad (35n)$$

$$C_{ii+1} = \frac{\alpha}{2} \qquad (35o)$$

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