Falling Water Tables in Horizontal/Sloping Aquifer

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Abstract: To describe falling water tables between two drains lying on a horizontal/sloping impermeable barrier, analytical solutions of the Boussinesq equation linearized by Baumann's and Werner's methods and numerical solutions of the nonlinear form of the Boussinesq equation using finite-difference and finite-element methods were obtained. A hybrid finite analytic method, in which the nonlinear Boussinesq equation was locally linearized and solved analytically after approximating the unsteady term by a simple finite-difference formula to approximately preserve the overall nonlinear effect by the assembly of locally analytic solutions, was also used to obtain a solution of the Boussinesq equation. Midpoints of falling water tables between two drains in a horizontal/sloping aquifer as obtained from various solutions were compared with already existing experimental values. Euclidean $L2$ and Tchebycheff L_∞ norms were used to rank the performance of various solutions with respect to experimental data. It was observed that the performance of the hybrid finite analytic solution is the best, followed by finite element, finite difference, analytical with Werner's linearization method, and analytical with Baumann's linearization method, respectively.

INTRODUCTION

Most of the subsurface drainage theories related to flatlands or moderately sloping lands have been developed by obtaining the solution of partial differential equations derived by Boussinesq (1877, 1904), based on the principle of continuity and Dupuit-Forchheimer assumptions. Many investigators, such as Schmid and Luthin (1964), Guitjens and Luthin (1965), Chauhan et al. (1968), Childs (1971), Towner (1975), Jaiswal and Chauhan (1975), Singh and Jacob (1977), Chapman (1980), Sloan and Moore (1984), Yates et al. (1985), Sewa Ram and Chauhan (1987a,b), Fipps and Skaggs (1989), Shukla et al. (1990, 1999), Sanford et al. (1993), Brutsaert (1994), Pi and Hjelmfelt (1994), Kalaidzidou-Paikou et al. (1997), Koussis et al. (1998), Connell et al. (1998), and Steenhuis et al. (1999), have obtained analytical, numerical, or experimental solutions of linearized or nonlinear forms of the Boussinesq continuity equation to describe spatial and temporal variation of water tables in an aquifer resting on a sloping impermeable barrier. A brief review of these studies has been presented by Upadhyaya (1999) and Upadhyaya and Chauhan (2000). Most of the analytical solutions obtained are approximate because of the linearization of the governing equation. Numerical solutions of nonlinear equations are better than the analytical solutions of linearized equations. However, the accuracy of such solutions can be tested by comparing the results with experimental values considering the latter ones as the benchmark solution. The objective of this paper is to obtain analytical solutions of the linearized Boussinesq equation and numerical solutions (based on finite-difference, finite-element, and hybrid finite analytic techniques) of the nonlinear Boussinesq equation to describe falling water tables between two drains in a horizontal or moderately sloping aquifer and comparison of predicted water table elevations with the experimental results obtained through studies of Chauhan (1967) and Chauhan et al. (1968) simulating falling water tables for the horizontal and sloping cases on a vertical Hele-Shaw model.

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BOUNDARY-VALUE PROBLEM AND SOLUTIONS

While formulating the boundary-value problem for falling water tables between two conventional parallel drains, it is assumed that due to instantaneous recharge the water table has reached the land surface and it falls because of drainage of the aquifer. The nonlinear second-order partial differential equation derived by Boussinesq (1904) to describe falling water tables between two drains located in a sloping aquifer may be written

$$h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 - \alpha\left(\frac{\partial h}{\partial x}\right) = \frac{f}{K}\frac{\partial h}{\partial t} \qquad (1a)$$

and for a horizontal aquifer (considering $\alpha = 0$)

$$h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 = \frac{f}{K}\frac{\partial h}{\partial t} \qquad (1b)$$

Here h = height of water table above the impermeable barrier (L) at a distance x and time t; α = slope of the impermeable barrier having a small value such that $\alpha = \sin\alpha = \tan\alpha$; K = hydraulic conductivity (L T^{-1}); and f = drainable porosity of the aquifer.

Two techniques may be used to linearize the nonlinear equations [(1*a*) and (1*b*)]. The first technique, known as Baumann's technique takes $h(x, t) = D + \varepsilon(x, t)$, where D is a characteristic depth and $\varepsilon << D$, and by neglecting higher order in ε, eliminates $(\partial h/\partial x)^2$. The second technique is based on Werner's transformation, $z = h^2$, and linearizes equations by setting $(1/\sqrt{z})\cdot(\partial z/\partial t) = (1/D)\cdot(\partial z/\partial t)$. Here the characteristic depth $D = h_0/2$ or average depth of flow and h_0 is initial water table height above the impermeable barrier. After applying the above linearization techniques, (1*a*) corresponding to a sloping aquifer can be expressed as (2*a*) and (3*a*) whereas (1*b*) corresponding to a horizontal aquifer can be expressed as (2*b*) and (3*b*) as below

$$\frac{\partial^2 h}{\partial x^2} - 2s\left(\frac{\partial h}{\partial x}\right) = \frac{1}{a}\frac{\partial h}{\partial t}; \quad \frac{\partial^2 h}{\partial x^2} = \frac{1}{a}\frac{\partial h}{\partial t} \qquad (2a,b)$$

$$\frac{\partial^2 z}{\partial x^2} - 2s\left(\frac{\partial z}{\partial x}\right) = \frac{1}{a}\frac{\partial z}{\partial t}; \quad \frac{\partial^2 z}{\partial x^2} = \frac{1}{a}\frac{\partial z}{\partial t} \qquad (3a,b)$$

Here $a = KD/f$; and $s = \alpha/2D$. Initial shape of the water table may be assumed flat, parabolic, or elliptical, depending on soil characteristics. In this study, a flat initial water table near the land surface and zero water table at the drains (neglecting the effect of a seepage surface) have been considered. The definition sketch of the flow problem is given in Fig. 1. The initial and boundary conditions in mathematical terms corresponding to (1*a*), (1*b*), (2*a*), and (2*b*) may be written

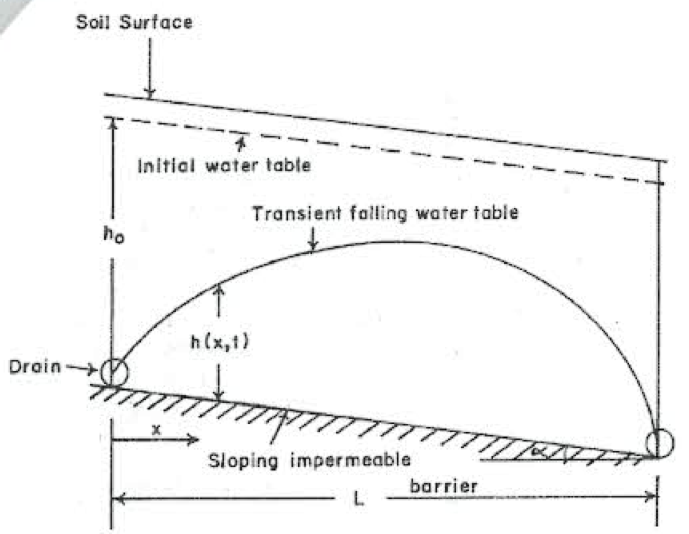

FIG. 1. Definition Sketch of Falling Water Table in Sloping Aquifer

$$h(x, 0) = h_0 \quad \text{at } t = 0 \quad \text{for } 0 < x < L \tag{4a}$$

$$h(0, t) = h(L, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = 0 \text{ and } x = L \tag{4b}$$

whereas, corresponding to (3*a*) and (3*b*), these conditions are

$$z(x, 0) = h_0^2 = z_0 \quad \text{at } t = 0 \quad \text{for } 0 < x < L \tag{5a}$$

$$z(0, t) = z(L, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = 0 \text{ and } x = L \tag{5b}$$

Here h_0 represents initial water table height above the impermeable barrier.

Analytical Solutions

An analytical solution of linearized Boussinesq equation (2*a*) with initial and boundary conditions (4*a*) and (4*b*) was obtained by devising a transformation that absorbs the term associated with slope and converts (2*a*) into a simple heat flow equation. The transformation may be given

$$h = ve^{sx - s^2at} \tag{6}$$

With this transformation the governing equation (2*a*) becomes

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{a}\frac{\partial v}{\partial t} \tag{7a}$$

and the initial and boundary conditions

$$v(x, 0) = h_0 e^{-sx} = f(x) \quad \text{at } t = 0 \quad \text{for } 0 < x < L \tag{7b}$$

$$v(0, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = 0 \tag{7c}$$

$$v(L, t) = 0 \quad \text{at } t > 0 \quad \text{for } x = L \tag{7d}$$

The solution of (7*a*) with initial and boundary conditions (7*b*)–(7*d*) is given by Ozisik (1980)

$$v(x, t) = \frac{2}{L}\sum_{m=1}^{\infty} e^{-a\beta_m^2 t} \sin \beta_m x \int_0^L f(x) \sin \beta_m x'\, dx' \tag{8}$$

with $\beta_m = m\pi/L$. Substitution of the value of $f(x)$ from (7*b*) in (8) and applying the inverse transformation gives the final solution

$$h(x, t) = \frac{2h_0}{L} e^{sx - s^2at} \sum_{m=1}^{\infty} e^{-a\beta_m^2 t} \sin \beta_m x \left\{\frac{(1 - (-1)^m e^{-sL})\beta_m}{(s^2 + \beta_m^2)}\right\} \tag{9}$$

If the aquifer is considered horizontal, the boundary-value problem is defined by (2*b*), (4*a*), and (4*b*) and the analytical solution may be obtained by putting $s = 0$ in (9)

$$h(x, t) = \frac{4h_0}{L} \sum_{m=1,3,5,\ldots}^{\infty} e^{-a\beta_m^2 t} \frac{\sin \beta_m x}{\beta_m} \tag{10}$$

This solution is similar to the solution proposed by Dumm (1954).

Analytical solutions of Boussinesq's equation linearized by Werner's transformation as given by (3*a*) and (3*b*) with initial and boundary conditions (5*a*) and (5*b*) were obtained using the same technique mentioned above. The transformation employed to (3*a*) is similar to (6) except that, in place of h, z is used. The transformed initial and boundary conditions are also the same as (7*b*)–(7*d*) except that in (7*b*), in place of h_0, z_0 is used. The analytical solution for falling water tables in a sloping aquifer in the form of $z(x, t)$ may be expressed

$$z(x, t) = \frac{2z_0}{L} e^{sx - s^2at} \sum_{m=1}^{\infty} e^{-a\beta_m^2 t} \sin \beta_m x \left\{\frac{(1 - (-1)^m e^{-sL})\beta_m}{(s^2 + \beta_m^2)}\right\} \tag{11}$$

and in case of a horizontal aquifer, corresponding to (3*b*), the analytical solution may be written

$$z(x, t) = \frac{4z_0}{L} \sum_{m=1,3,5,\ldots}^{\infty} e^{-a\beta_m^2 t} \frac{\sin \beta_m x}{\beta_m} \tag{12}$$

The solutions expressed by (11) and (12) are in the form of $z(x, t)$. To obtain the expression for $h(x, t)$, the square root of the expression for $z(x, t)$ is used.

Finite-Difference Solution

To obtain a finite-difference solution of nonlinear Boussinesq equation (1*a*) along with initial and boundary conditions given by (4*a*) and (4*b*), (1*a*) is made dimensionless with the help of a set of variables, $H = h/h_0$, $X = x/L$, and $T = Kh_0t/fL^2$. After transforming (1*a*) with these dimensionless variables, the governing equation and initial and boundary conditions may be written

$$H\frac{\partial^2 H}{\partial X^2} + \left(\frac{\partial H}{\partial X}\right)^2 - A\left(\frac{\partial H}{\partial X}\right) = \frac{\partial H}{\partial T} \tag{13}$$

or

$$\frac{1}{2}\frac{\partial^2 H^2}{\partial X^2} - A\left(\frac{\partial H}{\partial X}\right) = \frac{\partial H}{\partial T} \tag{14}$$

$$H(X, 0) = 1 \quad \text{at } T = 0 \quad \text{for } 0 < X < 1 \tag{15a}$$

$$H(0, T) = H(1, T) = 0 \quad \text{at } T > 0 \quad \text{for } X = 0 \text{ and } X = 1 \tag{15b}$$

where $A = \alpha L/h_0$.

Eq. (14) can be discretized in finite-difference form as below

$$\frac{H_m^{n+1} - H_m^n}{\Delta T} = \frac{1}{2(\Delta X)^2}[\theta(H_{m-1}^{n+1})^2 + (1-\theta)(H_{m-1}^n)^2 - 2\theta(H_m^{n+1})^2 - 2(1-\theta)(H_m^n)^2 + \theta(H_{m+1}^{n+1})^2 + (1-\theta)(H_{m+1}^n)^2] - \frac{A}{2\Delta X}[\theta(H_{m+1}^{n+1}) + (1-\theta)(H_{m+1}^n) - \theta(H_{m-1}^{n+1}) - (1-\theta)(H_{m-1}^n)] \tag{16}$$

Here θ may be assigned a value from 0 to 1, resulting in explicit or implicit finite-difference schemes. Subscript m denotes a variable in the space grid and subscript n denotes a variable in time.

Jain et al. (1994) proposed the following procedure to solve the system of nonlinear equations. Let $H_m^{n+1} = H_m^n + V_m^n$. Substituting this in the above equation one gets

$$V_m^n = \frac{\Delta T}{2(\Delta X)^2}[\theta(H_{m-1}^n + V_{m-1}^n)^2 + (1-\theta)(H_{m-1}^n)^2 - 2\theta(H_m^n + V_m^n)^2 - 2(1-\theta)(H_m^n)^2 + \theta(H_{m+1}^n + V_{m+1}^n)^2 + (1-\theta)(H_{m+1}^n)^2] - \frac{A\Delta T}{2\Delta X}[\theta(H_{m+1}^n + V_{m+1}^n) + (1-\theta)(H_{m+1}^n) - \theta(H_{m-1}^n + V_{m-1}^n) - (1-\theta)(H_{m-1}^n)] \quad (17)$$

Keeping $(\Delta T/\Delta X) = C$ and $(\Delta T/(\Delta X)^2) = \lambda$ and neglecting the terms of the order of $O(V^2)$, one gets the following equation:

$$V_{m-1}^n\left[\lambda\theta H_{m-1}^n + \frac{AC\theta}{2}\right] + V_m^n[-2\lambda\theta H_m^n - 1] + V_{m+1}^n\left[\lambda\theta H_{m+1}^n - \frac{AC\theta}{2}\right] = -\frac{\lambda}{2}[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] + \frac{AC}{2}[H_{m+1}^n - H_{m-1}^n] \quad (18)$$

By inputting different values of θ, one can obtain various finite-difference schemes.

If $\theta = 0$, one can get an explicit finite-difference scheme

$$V_m^n = \frac{\lambda}{2}[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] - \frac{AC}{2}[H_{m+1}^n - H_{m-1}^n] \quad (19a)$$

If $\theta = 0.5$, the Crank-Nicolson finite-difference scheme may be written

$$V_{m-1}^n\left[\lambda H_{m-1}^n + \frac{AC}{2}\right] - 2V_m^n[\lambda H_m^n + 1] + V_{m+1}^n\left[\lambda H_{m+1}^n - \frac{AC}{2}\right] = -\lambda[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] + AC[H_{m+1}^n - H_{m-1}^n] \quad (19b)$$

and if $\theta = 1.0$, one can obtain a fully implicit finite-difference scheme

$$V_{m-1}^n\left[\lambda H_{m-1}^n + \frac{AC}{2}\right] - V_m^n[2\lambda H_m^n + 1] + V_{m+1}^n\left[\lambda H_{m+1}^n - \frac{AC}{2}\right] = -\frac{\lambda}{2}[(H_{m-1}^n)^2 - 2(H_m^n)^2 + (H_{m+1}^n)^2] + \frac{AC}{2}[H_{m+1}^n - H_{m-1}^n] \quad (19c)$$

By putting $A = 0$ in (19a)–(19c), the expression of the finite-difference solution for falling water tables between two conventional level drains in a horizontal aquifer can be obtained.

This system of algebraic equations formed at a given time step is a tridiagonal matrix for which a solution can be obtained by any of the standard algorithms available in various texts of numerical analysis and V_{m-1}^n, V_m^n, and V_{m+1}^n can be computed. To get the values at $n + 1$ time step (i.e., H_{m-1}^{n+1}, H_m^{n+1}, and H_{m+1}^{n+1}), the values of H_{m-1}^n, H_m^n, and H_{m+1}^n are added into V_{m-1}^n, V_m^n, and V_{m+1}^n, respectively.

Finite-Element Solution

A finite-element solution of the dimensionless nonlinear Boussinesq equation as shown by (13) or (14) along with initial and boundary conditions (15a) and (15b) was obtained using Galerkin's method, the details of which are given in Pinder and Gray (1977). The flow domain is discretized as $0 = X_1 < X_2 < X_3 < X_4 < \cdots < X_{N-1} < X_N = 1$. Here N represents the number of nodes and $\Delta X = X_{i+1} - X_i$, where $i = 1, 2, 3, \ldots, N - 1 = M$, the number of elements.

To obtain the solution of Boussinesq's equation by Galerkin's method, a linear Lagrange polynomial is associated with each node. A typical basis function associated with each node X_i, as defined by Prenter (1975) is given below

$$N_i(X) = \frac{(X - X_{i-1})}{(X_i - X_{i-1})} \quad \text{for } X_{i-1} \le X \le X_i \quad (20a)$$

$$N_i(X) = \frac{(X_{i+1} - X)}{(X_{i+1} - X_i)} \quad \text{for } X_i \le X \le X_{i+1} \quad (20b)$$

The basis function $N_i(X)$ has the value unity at the node with which it is associated and is zero at all the other nodes. The basis function $N_i(X)$ has a component in each of the two elements joining at node X_i. Hence, over the elements (X_{i-1}, X_i) and (X_i, X_{i+1}) there are two nonzero basis functions $N_{i-1}(X)$, $N_i(X)$ and $N_i(X)$, $N_{i+1}(X)$. The basis function $N_i(X)$ over the elements (X_{i-1}, X_i) and (X_i, X_{i+1}) has already been defined by (20a) and (20b). The basis functions $N_{i-1}(X)$ over the element (X_{i-1}, X_i) and $N_{i+1}(X)$ over the element (X_i, X_{i+1}) are given below

$$N_{i-1}(X) = \frac{(X_i - X)}{(X_i - X_{i-1})} \quad \text{for } X_{i-1} \le X \le X_i \quad (20c)$$

$$N_{i+1}(X) = \frac{(X - X_i)}{(X_{i+1} - X_i)} \quad \text{for } X_i \le X \le X_{i+1} \quad (20d)$$

The value of all other basis functions are zero over the elements (X_{i-1}, X_i) and (X_i, X_{i+1}). The solution is approximated by $H^A(X, T)$ with the help of the basis functions as follows:

$$H^A(X, T) = \sum_{i=1}^{N} Z_i(T)N_i(X) \quad (21)$$

in which $Z_i(T)$ = unknown coefficients to be determined as a part of the solution. The multiplier $Z_i(T)$ associated with $N_i(X)$ at node i is the value of H at i. Because there are only two nonzero basis functions over an element (X_i, X_{i+1}), the summation is performed only over two consecutive indices, i and $i + 1$ to approximate the solution $H^A(X, T)$ over the element.

To carry out finite-element analysis, (13) or (14) may be written

$$L(H) = \frac{\partial}{\partial X}\left(H\frac{\partial H}{\partial X}\right) - A\left(\frac{\partial H}{\partial X}\right) - \frac{\partial H}{\partial T} = 0 \quad (22)$$

The expression $H^A(X, T)$ is an approximation for $H(X, T)$. Hence its substitution in (22) leaves a residual $L(H^A)$, which is used to determine the coefficients $Z_i(T)$. As there are N unknown coefficients to be determined, N constraints have to be imposed on the residual $L(H^A)$ to evaluate these coefficients. In Galerkin's finite-element method, the coefficients $Z_i(T)$ are determined by forcing the residual $L(H^A)$ to be orthogonal to the basis functions $N_i(X)$, where $i = 1, 2, 3, \ldots, N$. For this, the inner product of $L(H^A)$ with $N_i(X)$ has to be zero; i.e.

$$\langle L(H^A)\cdot N_i(X)\rangle = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \quad (23)$$

Substitution of (22) in (23) yields

$$\left\langle\frac{\partial}{\partial X}\left(H\frac{\partial H}{\partial X}\right), N_i(X)\right\rangle - \left\langle A\frac{\partial H}{\partial X}, N_i(X)\right\rangle - \left\langle\frac{\partial H}{\partial T}, N_i(X)\right\rangle = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \quad (24)$$

Hereafter for convenience H^A is written as H. Integration of (24) yields

$$\left\{H\frac{\partial H}{\partial X}\cdot N_i(X)\right\}\Bigg|_{X=0}^{X=1} - \int_0^1 \frac{d}{dX}N_i(X)\cdot\left(H\frac{\partial H}{\partial X}\right)dX - A\int_0^1 \frac{\partial H}{\partial X}\cdot N_i(X)\,dX - \int_0^1 \frac{\partial H}{\partial T}\cdot N_i(X)\,dX = 0 \quad \text{for } i = 1, 2, 3, \ldots, N \quad (25)$$

By substituting the value of H from (21) into (25), a system of N integral equations is obtained as below

$$\sum_{j=1}^{N}\int_0^1 N_i(X)N_j(X)\frac{dZ_j}{dT}\,dX+\frac{1}{2}\sum_{j=1}^{N}\int_0^1\frac{dN_i(X)}{dX}\frac{dN_j^2(X)}{dX}Z_j^2\,dX$$

$$+A\sum_{j=1}^{N}\int_0^1 N_i(X)\frac{dN_j(X)}{dX}Z_j\,dX=\left\{H\frac{\partial H}{\partial X}\cdot N_i(X)\right\}\Big|_{X=1}$$

$$-\left\{H\frac{\partial H}{\partial X}\cdot N_i(X)\right\}\Big|_{X=0}\quad \text{for } i=1,2,3,\ldots,N \tag{26}$$

or

$$\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)N_j(X)\frac{dZ_j}{dT}\,dX+\frac{1}{2}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e\frac{dN_i(X)}{dX}$$

$$\cdot\frac{dN_j^2(X)}{dX}Z_j^2\,dX+A\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)\frac{dN_j(X)}{dX}Z_j\,dX$$

$$=\left\{H\frac{\partial H}{\partial X}\right\}\Big|_{X=1}-\left\{H\frac{\partial H}{\partial X}\right\}\Big|_{X=0}\quad \text{for } i=1,2,3,\ldots,N \tag{27}$$

Eq. (27) can be rewritten

$$[G]\left\{\frac{dZ}{dT}\right\}+[B]\{Z^2\}+A[C]\{Z\}=\{F\} \tag{28}$$

where

$$[G]=G_{ij}=\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)N_j(X)\,dX \tag{29a}$$

$$[B]=B_{ij}=\frac{1}{2}\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e\frac{dN_i(X)}{dX}\frac{dN_j^2(X)}{dX}\,dX \tag{29b}$$

$$[C]=C_{ij}=\sum_{j=1}^{N}\sum_{e=1}^{M}\int_e N_i(X)\frac{dN_j(X)}{dX}\,dX \tag{29c}$$

$$\{F_i\}=0\quad \text{for } i=2,3,4,\ldots,N-1 \tag{29d}$$

$$\{F_1\}=-\left(H\frac{\partial H}{\partial X}\right)\Big|_{X=0};\quad \{F_N\}=\left(H\frac{\partial H}{\partial X}\right)\Big|_{X=1} \tag{29e,f}$$

Coefficient matrices were evaluated and are given in the Appendix.

Eq. (28) may be written in finite-difference form

$$[G]\left\{\frac{Z(T+\Delta T)-Z(T)}{\Delta T}\right\}+[B]\{Z^2(T+\Delta T)\}$$

$$+A[C]\{Z(T+\Delta T)\}=\{F(T)\} \tag{30}$$

Let $Z(T+\Delta T)=Z(T)+V(T)$. Substitution of this relationship in (30) yields

$$[G]\left\{\frac{V(T)}{\Delta T}\right\}+[B]\{Z^2(T)+2Z(T)V(T)+V^2(T)\}$$

$$+A[C]\{Z(T)+V(T)\}=\{F(T)\} \tag{31}$$

Neglecting the terms of $O(V^2(T))$ gives

$$[G]\{V(T)\}+\Delta T[B]\{2Z(T)V(T)\}+\Delta TA[C]\{V(T)\}$$

$$=-\Delta T[B]\{Z^2(T)\}-\Delta TA[C]\{Z(T)\}+\Delta T\{F(T)\} \tag{32}$$

or

$$[[G]+2\Delta T[B]\{Z(T)\}+\Delta TA[C]]\{V(T)\}$$

$$=-\Delta T[B]\{Z^2(T)\}-\Delta TA[C]\{Z(T)\}+\Delta T\{F(T)\} \tag{33}$$

The solution of this system of algebraic equations provides the values of $V(T)$ at different nodes. This $V(T)$ value at a particular node is added to the value of $Z(T)$ at that node to get the value of $Z(T+\Delta T)$ at that particular node for the next time step. To obtain the solution for the water table profile in horizontal aquifers, A is substituted as zero. The resulting expression will be the finite-element solution for falling water tables between two conventional level drains in a horizontal aquifer.

Hybrid Finite Analytic Solution

Pi and Hjelmfelt (1994) solved an extended Dupuit-Forchheimer equation to describe water table profiles and lateral subsurface streamflow in a sloping aquifer using a hybrid finite analytic method based on the approach of Chen (1988). A local linearized 1D Dupuit-Forchheimer equation was solved analytically in space and discretized in time by a simple difference formula. The resultant system of algebraic equations approximated the overall nonlinear effect because the coefficient of diffusion and the term $(\partial h/\partial x)^2$ were treated as constants only in the local regions. A four-point numerical formula provided stable and sufficiently accurate results with simple calculations and without small time steps. Steady-state profiles of water table and lateral subsurface storm flow obtained from their study compared well with the results of previous investigators.

Using the approach of Chen (1988), the hybrid finite analytic solution of nonlinear nondimensionalized equations (13) or (14), along with initial and boundary conditions defined by (15a) and (15b), was obtained. The procedure for obtaining hybrid finite analytic solutions to describe falling water tables between two conventional level drains is as follows.

If H associated with $(\partial^2H/\partial X^2)$ in (13) is replaced by H_a, the dimensionless average depth of flow, it may be written

$$\frac{\partial^2 H}{\partial X^2}+\frac{1}{H_a}\left(\frac{\partial H}{\partial X}\right)^2-\frac{A}{H_a}\left(\frac{\partial H}{\partial X}\right)=\frac{1}{H_a}\frac{\partial H}{\partial T} \tag{34}$$

Assuming the terms $1/H_a(\partial H/\partial X)^2$, $1/H_a(\partial H/\partial T)$, and A/H_a as constants denoted by C, E, and $2s_1$, respectively, in a small subregion and performing integration, one gets the following equation:

$$\frac{dH}{dX}-2s_1H=(E-C)X+F \tag{35}$$

The solution of this first-order ordinary differential equation is

$$H(X)=Ge^{2s_1X}-(E-C)\frac{X}{2s_1}+I \tag{36}$$

Discretization of (36) in space and time yields the following equations:

$$H_{i-1}^{n+1}=Ge^{-2s_1\Delta X}+(E-C)\frac{\Delta X}{2s_1}+I \tag{37a}$$

$$H_{i+1}^{n+1}=Ge^{2s_1\Delta X}-(E-C)\frac{\Delta X}{2s_1}+I \tag{37b}$$

$$H_i^{n+1}=G+I \tag{37c}$$

because H_i^{n+1} represents the point where $\Delta X=0$.

Simplification of these equations yield a tridiagonal matrix

$$A_iH_{i-1}^{n+1}+B_iH_i^{n+1}+C_iH_{i+1}^{n+1}=D_iH_i^n+E_i \tag{38}$$

where

$$A_i=-\frac{e^{s_1\Delta X}}{e^{s_1\Delta X}+e^{-s_1\Delta X}};\quad B_i=1+\frac{\Delta X}{2H_as_1\Delta T}\tanh(s_1\Delta X) \tag{39a,b}$$

$$C_i=-\frac{e^{-s_1\Delta X}}{e^{s_1\Delta X}+e^{-s_1\Delta X}};\quad D_i=\frac{\Delta X}{2H_as_1\Delta T}\tanh(s_1\Delta X) \tag{39c,d}$$